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THEORY OF FINITE SYSTEMS OF PARTICLES

I. THE GREEN FUNCTION

BY

CLASINE VAN WINTER

København 1964
Kommissionær: Ejnar Munksgaard

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Synopsis

A study is made of the resolvent operator for a system consisting of any finite number of particles. It is assumed that all the forces in the system arise from square-integrable local two-body interactions. The resolvent is considered for complex energies not in the continuous spectrum of the Hamiltonian. It is represented by an integral operator, the kernel of which is called the Green function. In the energy plane cut along the positive real axis, the Green function $G^{(2)}$ for a system of two particles is obtained from an integral equation. Owing to the assumption on the interaction, this equation has a Hilbert-Schmidt kernel. As a result it can be solved by the Fredholm method. With $G^{(2)}$ as a supposedly known quantity, an integral equation for $G^{(3)}$ is constructed which again has a Hilbert-Schmidt kernel, and hence is again soluble. Next the procedure is extended to successively larger systems. For n particles, an integral equation is found from which the Green function $G^{(n)}$ can be obtained once the functions $G^{(2)}, G^{(3)}, \ldots, G^{(n-1)}$ are known. By an induction argument an upper bound is derived for the Schmidt-norm of each successive kernel $K^{(n)}$. This is shown to be finite at every interior point of the energy plane cut along the continuous spectrum of the n-particle Hamiltonian $H^{(n)}$. In the cut plane the Fredholm expressions converge irrespective of the strength of the interaction. The Green function $G^{(n)}$ is regular except for the cut and for possible poles on the negative real axis. These correspond to bound states, the eigenfunctions of $H^{(n)}$ following directly from the residues of $G^{(n)}$. The present method does not give information on the behaviour of the Green function in the neighbourhood of the continuous spectrum. This is the subject of a forthcoming paper on the theory of scattering.

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1.1. Introduction

This is the first of a series of papers concerned with various general properties of systems of n particles governed by Hamiltonians of the form

$$H'(\boldsymbol{X}) = -\sum_{i=1}^{n} \frac{1}{2\,m_i} \Delta(\boldsymbol{X}_i) + \sum_{i<j} V_{ij}(\boldsymbol{X}_i - \boldsymbol{X}_j). \tag{1.1.1}$$

In eq. (1.1.1) $\boldsymbol{X}_i = (X_{i1}, X_{i2}, X_{i3})$ is the space coordinate of particle i. The three components of $\boldsymbol{X}_i$ range from $-\infty$ to ∞. The system is not enclosed in a finite box, neither are periodic boundary conditions imposed. The symbol Δ stands for the Laplace operator.

The masses of the particles are denoted by m_i. It is assumed that $m_i \neq 0$. The various masses may or may not be equal. The particles are treated as distinguishable. It has been checked that the results of the present paper allow a simple specialization in the case of identical particles, but this point is not pursued here.

In each particular system, the number of particles is kept fixed. No use is made of creation and destruction operators and the techniques associated with these. There are thus distinct problems for $n = 1, 2, 3, \ldots$. Only finite values of n are considered.

The two-body interaction V_{ij} is taken to be real. It is assumed that

$$\int_{-\infty}^{\infty} [V_{ij}(\boldsymbol{X})]^2 d^3\boldsymbol{X} < \infty. \tag{1.1.2}$$

This means, roughly, that at the origin V_{ij} must be less singular than $|\boldsymbol{X}|^{-\frac{3}{2}}$, while at large distances it must tend to zero faster than $|\boldsymbol{X}|^{-\frac{3}{2}}$. The Coulomb interaction is therefore excluded, as are potentials with hard cores.

It is an important point that the relation (1.1.2) does not imply an assumption on the strength of the potential. The results of this and the following papers apply to weak and strong interactions alike, as they are based on methods which go beyond perturbation theory.

In tackling the problem of n particles, the most straightforward starting-point would seem to be the Schrödinger equation

$$H'(\boldsymbol{X})\,\Psi(\boldsymbol{X}) = E\,\Psi(\boldsymbol{X}). \tag{1.1.3}$$

Solving this for Ψ, one would expect that in the case of negative energy only certain discrete values of E are admitted, while for positive E the equation would admit of

a continuum of solutions which satisfy certain outgoing boundary conditions and thereby describe scattering states in which in suitable regions of configuration space the system behaves asymptotically as an assembly of free subsystems. It is of course known that this program works quite well for a system of two particles with spherically symmetric interaction. It must be borne in mind, however, that in this case the success of the method is due to the Schrödinger equation being separable into ordinary differential equations for which methods of solution are readily available. As soon as one passes on to three or more particles, one is always left with partial differential equations. Now, the properties of such equations are quite different from those of ordinary differential equations. As a matter of fact, one does not even have an existence theorem to guarantee that for more-particle systems the desired scattering solutions can actually be found.

It is for this reason that we do not want to go into the question of scattering wave-functions. In the following the Hamiltonian (1.1.1) is considered as an operator which acts on square-integrable functions of $\boldsymbol{X}$, and the n-body problem is approached from the point of view of the theory of Hilbert space. This does not only make it possible to put the formalism on a firm mathematical basis, it also makes available some very powerful computational methods.

One of our principal tools is the resolvent of the Hamiltonian and the Green function G associated with it. As is usual, the resolvent is studied as an analytic function of the energy in the complex plane cut along the real axis from a point M to ∞. At every interior point of the cut plane, we find the Green function for any finite number of particles. To achieve this, we observe first of all that in the cut plane the two-particle Green function $G^{(2)}$ satisfies an integral equation with a Hilbert-Schmidt kernel. Since such an equation is known to be soluble by the methods of the Fredholm theory of integral equations, we can find $G^{(2)}$. It takes the form of a quotient of two series, which are convergent owing to the assumption (1.1.2) that the interactions V_{ij} are square-integrable. The Green function $G^{(2)}$ is now used as a supposedly known quantity in an integral equation for the three-particle function $G^{(3)}$. The point is here that this equation is set up in such a way that it has a Hilbert-Schmidt kernel once again. As a result of this, it can be solved rigorously by the same method as that applied to $G^{(2)}$. Next, $G^{(2)}$ and $G^{(3)}$ are entered into an equation for $G^{(4)}$, and so on. Summarizing, there is an iteration procedure whereby the Green function $G^{(n)}$ for any number of particles can be found once the functions $G^{(m)}$ for $m = 2, 3, \ldots, n-1$ are known. This method of constructing the Green function leads in a natural way to the energies and the wave-functions of bound states. It forms the main contents of the present paper.

In forthcoming papers we shall outline a general theory of scattering, with special emphasis on the channel concept and on sufficient conditions on the interaction to guarantee that a useful scattering formalism can indeed be developed. The next step consists in defining the S-operator, and expressing it in terms of the Green function. The most intricate point to be discussed in this connection concerns the limiting

properties of the Green function as the energy tends to the cut in the complex plane. This limiting behaviour will be shown to be intimately connected with the asymptotic properties of wave-packets at large times, and with the falling-off of the interaction at large distances.

Up to this stage, the formalism is completely general, being capable of describing scattering and reactions among any number of particles or bound fragments. The remaining part of the investigation is devoted to the special case in which both in the distant past and in the remote future the total scattering system is split up into not more than two subsystems. For this case, results will be presented on the analytic properties of scattering amplitudes for fixed momentum transfer, and on dispersion relations. This part of the work also gives information on the asymptotic behaviour of wave-functions for bound states. The last item to be discussed is the scattering of partial waves, for which a new type of resonance formula will be given. The principal feature is here that, along with the well-known Breit-Wigner behaviour in the neighbourhood of an isolated resonance, we find an energy-dependent scattering radius which through a dispersion relation is connected with the absorption.

1.2. The Hamiltonian

1.2.1. Coordinates

In treating the Hamiltonian (1.1.1) it is useful to go over from the coordinates $\boldsymbol{X}$ to new coordinates $\boldsymbol{x}$ so that the energy of the centre-of-mass motion is separated off. This can easily be achieved in various ways, one possibility being the choice(1)

$$\begin{aligned} \boldsymbol{x}_k &= \left(\frac{2\,m_{k+1}}{M_k\,M_{k+1}}\right)^{\frac{1}{2}} \sum_{j=1}^{k} m_j(\boldsymbol{X}_{k+1}-\boldsymbol{X}_j) \quad (k=1,\,2,\,\ldots,\,n-1), \\ \boldsymbol{x}_n &= \frac{\sqrt{2}}{\sqrt{M_n}} \sum_{j=1}^{n} m_j \boldsymbol{X}_j, \\ M_k &= \sum_{j=1}^{k} m_j. \end{aligned} \tag{1.2.1}$$

With this choice of coordinates, the kinetic-energy operator takes the form

$$-\sum_{i=1}^{n} \frac{1}{2\,m_i} \Delta(\boldsymbol{X}_i) = -\sum_{i=1}^{n} \Delta(\boldsymbol{x}_i), \tag{1.2.2}$$

no mixed derivatives appearing. It follows from eq. (1.2.1) that

$$\boldsymbol{X}_i - \boldsymbol{X}_j = \sum_{k=1}^{j-1} c_{ij}^{k} \boldsymbol{x}_k \quad (i<j) \tag{1.2.3}$$

with certain constants c depending on the masses of the particles.

In eq. (1.2.1), $\boldsymbol{x}_n$ is the coordinate of the centre of mass, apart from a scale factor. The coordinates $\boldsymbol{x}_k$ ($k = 1, 2, \ldots, n-1$) describe the internal motion. The Hamiltonian of the internal motion takes the form

$$H'(\boldsymbol{x}) = -\sum_{i=1}^{n-1} \Delta(\boldsymbol{x}_i) + \sum_{i<j} V_{ij}\left(\sum_{k=1}^{j-1} c_{ij}^k \boldsymbol{x}_k\right). \tag{1.2.4}$$

Here, $\boldsymbol{x}_1$ is proportional to the distance between particles 2 and 1, $\boldsymbol{x}_2$ to the distance between particle 3 and the centre of mass of 2 and 1, and so on. It is easily checked that an alternative system without mixed derivatives in the kinetic energy can be obtained as follows(1). First the system of n particles is split into k subsystems. In each subsystem internal coordinates are introduced according to eq. (1.2.1). Next, a coordinate is used for the distance between the centres of mass of the groups 2 and 1, then a coordinate for the distance between the centres of mass of group 3 and the combined groups 2 and 1, and so on. This takes $n-k$ internal coordinates and $k-1$ relative coordinates between the k groups, so that just one coordinate is left for the centre of mass of the system as a whole. In the following we use all the coordinate systems of this sort, as the need arises.

1.2.2. Self-adjointness

We now want to consider $H'(\boldsymbol{x})$ as an operator in the Hilbert space of square-integrable functions of $\boldsymbol{x}$. This space is denoted by $\mathfrak{L}^2$. In cases of ambiguity we may occasionally write $\mathfrak{L}^2(\boldsymbol{x})$. For the norm and the inner product in $\mathfrak{L}^2$ we use the notations

$$\|f\| = \left[\int |f(\boldsymbol{x})|^2 d\boldsymbol{x}\right]^{\frac{1}{2}}, \quad (g, f) = \int \bar{g}(\boldsymbol{x}) f(\boldsymbol{x})\, d\boldsymbol{x}. \tag{1.2.5}$$

It is assumed that the reader is familiar with the basic ideas of the theory of Hilbert space, which can be found in the textbooks by ACHIESER and GLASMANN(2), RIESZ and SZ.-NAGY(3), STONE(4), and others.

To get the full benefit of the techniques of Hilbert space, we must define our Hamiltonian more carefully than is implied by eq. (1.2.4). As it stands there, it is too vague, because there does not seem to be a satisfactory criterion for determining the set of functions in $\mathfrak{L}^2$ on which a differential operator can be allowed to act. The way out of this difficulty was indicated by KATO(5) and STUMMEL(6). First the domain of the differential operator was restricted to a dense set in $\mathfrak{L}^2$ consisting of smooth functions which at infinity tend to zero sufficiently rapidly. The operator thus defined is hermitian, but it is not self-adjoint. It was shown, however, that under certain assumptions on the interaction it is essentially self-adjoint, i.e. that it has one and only one self-adjoint extension. It is this self-adjoint extension which we take as our Hamiltonian. It is denoted by H or by $H(\boldsymbol{x})$. The domain of H is denoted by $\mathfrak{D}(H)$.

Let us write the differential operator $H'(\boldsymbol{x})$ as the sum of the kinetic energy and the interaction,

$$H'(\boldsymbol{x}) = H_0'(\boldsymbol{x}) + V(\boldsymbol{x}). \tag{1.2.6}$$

Then, a sufficient condition for essential self-adjointness given by STUMMEL(6) is

$$\left.\begin{aligned} \int_{|\boldsymbol{x}|\leq 1} [V(\boldsymbol{x}-\boldsymbol{y})]^2 \, |\boldsymbol{x}|^{-3n+7-\alpha} \, d^3\boldsymbol{x}_1 \, d^3\boldsymbol{x}_2 \ldots d^3\boldsymbol{x}_{n-1} < C, \\ |\boldsymbol{x}| = (|\boldsymbol{x}_1|^2 + |\boldsymbol{x}_2|^2 + \ldots + |\boldsymbol{x}_{n-1}|^2)^{\frac{1}{2}}, \end{aligned}\right\} \tag{1.2.7}$$

for every $\boldsymbol{y}$, some constant C, and a suitable $\alpha > 0$. This is clearly fulfilled in the present case. For let us take V_{12}. If $n \geq 3$, it satisfies

$$\left.\begin{aligned} \int_{|\boldsymbol{x}|\leq 1} [V_{12}(c_{12}^1(\boldsymbol{x}_1-\boldsymbol{y}_1))]^2 \, |\boldsymbol{x}|^{-3n+7-\alpha} \, d^3\boldsymbol{x}_1 \, d^3\boldsymbol{x}_2 \ldots d^3\boldsymbol{x}_{n-1} \\ \leq \int_{-\infty}^{\infty} [V_{12}(c_{12}^1(\boldsymbol{x}_1-\boldsymbol{y}_1))]^2 \, d^3\boldsymbol{x}_1 \int_{|\boldsymbol{x}_2|^2+\ldots+|\boldsymbol{x}_{n-1}|^2 \leq 1} (|\boldsymbol{x}_2|^2+\ldots+|\boldsymbol{x}_{n-1}|^2)^{-\frac{3}{2}n+\frac{7}{2}-\frac{\alpha}{2}} d^3\boldsymbol{x}_2 \ldots d^3\boldsymbol{x}_{n-1} < C_{12} \end{aligned}\right\} \tag{1.2.8}$$

provided $\alpha < 1$, by virtue of the condition (1.1.2) on V_{ij}. If $n = 2$, the same holds true by an even simpler argument. By a permutation among the coordinates, similar bounds are readily derived for the remaining V_{ij}. The final result then follows from Schwarz's inequality.

It follows from the paper by KATO(5) that $H_0'(\boldsymbol{x})$ has a unique self-adjoint extension H_0. If $\mathfrak{D}(H_0)$ stands for the domain of H_0, it was shown that for every f in $\mathfrak{D}(H_0)$ the quantity Vf belongs to $\mathfrak{L}^2$. It was also shown that

$$\mathfrak{D}(H) = \mathfrak{D}(H_0), \quad H = H_0 + V. \tag{1.2.9}$$

1.2.3. The resolvent

The importance of the Hamiltonian being self-adjoint derives from the fact that this implies that it has a resolvent and a spectral resolution. To the latter concept we come back later. Our first object is the resolvent $R(\lambda)$,

$$R(\lambda) = (H-\lambda)^{-1}, \tag{1.2.10}$$

where λ is a complex number. The resolvent is defined for the set of points λ for which it exists and is a bounded linear operator with domain $\mathfrak{L}^2$. This certainly includes all non-real λ (ACHIESER and GLASMANN(2) section 43). It satisfies (ref.(2) section 44)

$$R(\lambda) - R(\mu) = (\lambda-\mu) \, R(\lambda) \, R(\mu) = (\lambda-\mu) \, R(\mu) \, R(\lambda), \tag{1.2.11}$$

$$R^*(\lambda) = R(\bar{\lambda}), \tag{1.2.12}$$

where the asterisk denotes the adjoint of an operator, and the bar the complex conjugate of a number.

Since $R(\lambda)f$ belongs to $\mathfrak{D}(H)$ whenever f belongs to $\mathfrak{L}^2$, eq. (1.2.9) shows that $H_0R(\lambda)$ and $VR(\lambda)$ are bounded operators. It is therefore justified to write

$$VR(\lambda) = (\lambda - H_0)R(\lambda) + 1. \tag{1.2.13}$$

Applying $R_0(\lambda)$, the resolvent of H_0, to both sides yields the equation

$$R_0(\lambda)VR(\lambda) = R_0(\lambda) - R(\lambda). \tag{1.2.14}$$

It is the main purpose of the present paper to show that in the case under investigation the resolvent is an integral operator,

$$R(\lambda)f(\boldsymbol{x}) = \int G(\boldsymbol{x}, \boldsymbol{y}; \lambda) f(\boldsymbol{y})\, d\boldsymbol{y} \tag{1.2.15}$$

for every f in $\mathfrak{L}^2$ and almost every $\boldsymbol{x}$, and to determine the kernel G. The function G is called the Green function. This agrees with the Green-function concept as used by COURANT and HILBERT(7), and more recently by TITCHMARSH(8). Loosely speaking, G satisfies

$$(H - \lambda)\, G(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \delta(\boldsymbol{x} - \boldsymbol{y}). \tag{1.2.16}$$

In finding the Green function $G^{(n)}$ for n particles, i.e. $(3n-3)$-dimensional $\boldsymbol{x}$ and $\boldsymbol{y}$ $(n = 2, 3, \ldots)$, our starting-point is the fact that in the case of no interaction it is known to exist and to have the form (TITCHMARSH(8) section 13.7)

$$G_0^{(n)}(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \frac{i}{4}\left[\frac{\sqrt{\lambda}}{2\pi|\boldsymbol{x}-\boldsymbol{y}|}\right]^{\frac{3}{2}n-\frac{5}{2}} H^{(1)}_{\frac{3}{2}n-\frac{5}{2}}(\sqrt{\lambda}\,|\boldsymbol{x}-\boldsymbol{y}|) \quad (n = 2, 3, \ldots), \tag{1.2.17}$$

$H^{(1)}$ being the Hankel function of the first kind. If $n = 2$, this reduces to

$$G_0^{(2)}(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} e^{i\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}. \tag{1.2.18}$$

Here and in the following, λ must always be restricted to

$$0 < \arg\lambda < 2\pi, \quad \operatorname{Im}\sqrt{\lambda} > 0. \tag{1.2.19}$$

1.3. The Green function for two particles

1.3.1. Method of construction

In determining the Green function, it is useful to start with two particles. The methods to be developed for this case are shown later to apply to larger numbers of particles without essential modification. Also, the results are required for the general n-particle problem, as our approach to this involves an iteration procedure.

For two particles, the relative motion depends on a three-dimensional vector $\boldsymbol{x}$. The interaction consists of a single term which, for simplicity, is denoted by $V(\boldsymbol{x})$. In the present section, all superscripts referring to the number of particles are dropped.

If f is any function in $\mathfrak{L}^2$, the functions

$$h(\boldsymbol{x};\lambda) = R(\lambda)f(\boldsymbol{x}), \quad h_0(\boldsymbol{x};\lambda) = R_0(\lambda)f(\boldsymbol{x}) = \int G_0(\boldsymbol{x},\boldsymbol{y};\lambda)f(\boldsymbol{y})\,d\boldsymbol{y} \tag{1.3.1}$$

obviously belong to $\mathfrak{L}^2$. From the resolvent equation (1.2.14) together with our knowledge of $R_0(\lambda)$ contained in eqs. (1.2.15) and (1.2.18), it follows that

$$h(\boldsymbol{x};\lambda) = h_0(\boldsymbol{x};\lambda) - \frac{1}{4\pi}\int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|}\, e^{i\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}\, V(\boldsymbol{y})\, h(\boldsymbol{y};\lambda)\,d\boldsymbol{y}. \tag{1.3.2}$$

Denoting the kernel of this integral equation by K,

$$K(\boldsymbol{x},\boldsymbol{y};\lambda) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\, e^{i\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}\, V(\boldsymbol{y}), \tag{1.3.3}$$

we observe the important fact that, if $\operatorname{Im}\sqrt{\lambda} > 0$,

$$\left.\begin{aligned} \iint |K(\boldsymbol{x},\boldsymbol{y};\lambda)|^2 d\boldsymbol{x}\,d\boldsymbol{y} &= \frac{1}{16\pi^2}\int [V(\boldsymbol{y})]^2 d\boldsymbol{y} \int \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^2}\, |e^{i\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}|^2 d\boldsymbol{x} \\ &= \frac{1}{8\pi \operatorname{Im}\sqrt{\lambda}} \int [V(\boldsymbol{y})]^2 d\boldsymbol{y} < \infty, \end{aligned}\right\} \tag{1.3.4}$$

owing to the assumption that V is square-integrable, eq. (1.1.2). Kernels which are square-integrable in the sense of eq. (1.3.4) are known as completely continuous, or Hilbert-Schmidt kernels. In the following we denote the set of all Hilbert-Schmidt kernels by $\mathfrak{L}^2$ or $\mathfrak{L}^2(\boldsymbol{x})$.

The importance of this set derives from the fact that for integral equations with kernels in $\mathfrak{L}^2$ the solution is known. As a matter of fact, it follows from the works of CARLEMAN(9), SMITHIES(10), and MICHLIN(11) that with minor alterations the Fredholm theory can be made applicable to these equations (see also SMITHIES(12) ch. VI, and ZAANEN(13) ch. 9, section 17).

To make this clear, let us define

$$\Delta(\lambda) = \sum_{p=0}^{\infty} \Delta_p(\lambda), \quad D(\boldsymbol{x},\boldsymbol{y};\lambda) = \sum_{p=0}^{\infty} D_p(\boldsymbol{x},\boldsymbol{y};\lambda), \tag{1.3.5}$$

where

$$\left.\begin{aligned} \Delta_0(\lambda) &= 1, \quad \Delta_1(\lambda) = 0, \quad D_0(\boldsymbol{x},\boldsymbol{y};\lambda) = K(\boldsymbol{x},\boldsymbol{y};\lambda), \\ \Delta_p(\lambda) &= -\frac{1}{p}\iint K(\boldsymbol{x},\boldsymbol{y};\lambda)\, D_{p-2}(\boldsymbol{y},\boldsymbol{x};\lambda)\,d\boldsymbol{x}\,d\boldsymbol{y} \quad (p \geq 2), \\ D_p(\boldsymbol{x},\boldsymbol{y};\lambda) &= \Delta_p(\lambda)\, K(\boldsymbol{x},\boldsymbol{y};\lambda) + \int K(\boldsymbol{x},\boldsymbol{z};\lambda) D_{p-1}(\boldsymbol{z},\boldsymbol{y};\lambda)\,d\boldsymbol{z} \quad (p \geq 1). \end{aligned}\right\} \tag{1.3.6}$$

It is then shown in the above references that if h_0 belongs to $\mathfrak{L}^2$ and λ is such that $\Delta(\lambda) \neq 0$, the equation (1.3.2) with the kernel (1.3.3) has one and only one solution in $\mathfrak{L}^2$, which takes the form

$$h(\boldsymbol{x};\lambda) = h_0(\boldsymbol{x};\lambda) + \frac{1}{\Delta(\lambda)} \int D(\boldsymbol{x},\boldsymbol{y};\lambda)\, h_0(\boldsymbol{y};\lambda)\, d\boldsymbol{y}. \tag{1.3.7}$$

If for any kernel in $\mathfrak{L}^2$ we denote the $\mathfrak{L}^2$-norm by

$$|K| = \left[\iint |K(\boldsymbol{x},\boldsymbol{y})|^2\, d\boldsymbol{x}\, d\boldsymbol{y}\right]^{\frac{1}{2}}, \tag{1.3.8}$$

we have

$$|\Delta_p(\lambda)| \leq (e/p)^{\frac{1}{2}p} |K(\lambda)|^p \quad (p \geq 1). \tag{1.3.9}$$

This clearly shows that the series for Δ is convergent for all values of λ for which $|K(\lambda)|$ is finite, i.e. for all λ with $\operatorname{Im} \sqrt{\lambda} > 0$ (cf. eq. (1.3.4)). It is to be noted that the convergence properties are independent of the absolute magnitude of the interaction V. As for the series for D, it is known that each kernel D_p belongs to $\mathfrak{L}^2$,

$$|D_p(\lambda)| \leq \sqrt{e}\, (e/p)^{\frac{1}{2}p} |K(\lambda)|^{p+1} \quad (p \geq 1). \tag{1.3.10}$$

From this it follows that D also belongs to $\mathfrak{L}^2$, and that the series for D converges in mean square.

In view of the expressions (1.3.1) for h and h_0, equation (1.3.7) very strongly suggests that

$$R(\lambda) f(\boldsymbol{x}) = \int G_0(\boldsymbol{x},\boldsymbol{z};\lambda) f(\boldsymbol{z})\, d\boldsymbol{z} + \frac{1}{\Delta(\lambda)} \int f(\boldsymbol{z})\, d\boldsymbol{z} \int D(\boldsymbol{x},\boldsymbol{y};\lambda)\, G_0(\boldsymbol{y},\boldsymbol{z};\lambda)\, d\boldsymbol{y}. \tag{1.3.11}$$

If we can justify this, we have thereby established that the resolvent is an integral operator, and we have found an expression for its kernel, the Green function G,

$$G(\boldsymbol{x},\boldsymbol{z};\lambda) = G_0(\boldsymbol{x},\boldsymbol{z};\lambda) + \frac{1}{\Delta(\lambda)} \int D(\boldsymbol{x},\boldsymbol{y};\lambda)\, G_0(\boldsymbol{y},\boldsymbol{z};\lambda)\, d\boldsymbol{y}. \tag{1.3.12}$$

The distinction between eqs. (1.3.7) and (1.3.11) is that in eq. (1.3.11) the integration with respect to $\boldsymbol{y}$ is performed first, while in eq. (1.3.7) this is done last. The problem is therefore to prove that in eq. (1.3.11) the integrations may be interchanged. To show this, let us first take any f and g in $\mathfrak{L}^2$. We know from eq. (1.2.12) that

$$(g, R(\lambda) f) = (R^*(\lambda) g, f) = (R(\bar{\lambda}) g, f), \tag{1.3.13}$$

so that

$$\left.\begin{aligned} \int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int G_0(\boldsymbol{x},\boldsymbol{y};\lambda) f(\boldsymbol{y})\, d\boldsymbol{y} &= \int f(\boldsymbol{y})\, d\boldsymbol{y} \int \overline{G}_0(\boldsymbol{y},\boldsymbol{x};\bar{\lambda})\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int f(\boldsymbol{y})\, d\boldsymbol{y} \int G_0(\boldsymbol{x},\boldsymbol{y};\lambda)\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x}, \end{aligned}\right\} \tag{1.3.14}$$

where use has been made of the symmetry properties

$$G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda) = G_0(\boldsymbol{y}, \boldsymbol{x}; \lambda), \tag{1.3.15}$$

$$\overline{G}_0(\boldsymbol{x}, \boldsymbol{y}; \lambda) = G_0(\boldsymbol{x}, \boldsymbol{y}; \bar{\lambda}), \tag{1.3.16}$$

which follow directly from the known expression for G_0.

Equation (1.3.14) shows that with straightforward functions f and g the order of integration may indeed be inverted. For the more complicated case presented by eq. (1.3.11), we note that according to Fubini's theorem (BURKILL(14) section 5.4) $D(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ is square-integrable with respect to $\boldsymbol{y}$ for almost every $\boldsymbol{x}$. The integrations may therefore be interchanged also in this case, for almost every $\boldsymbol{x}$. From this observation the desired equation (1.3.12) for the Green function now follows, as a set of points of measure zero is immaterial in a Hilbert-space formalism.

1.3.2. Symmetry

It is clear from the method of construction that

$$\overline{G}(\boldsymbol{x}, \boldsymbol{y}; \lambda) = G(\boldsymbol{x}, \boldsymbol{y}; \bar{\lambda}). \tag{1.3.17}$$

We now want to show that

$$G(\boldsymbol{x}, \boldsymbol{y}; \lambda) = G(\boldsymbol{y}, \boldsymbol{x}; \lambda). \tag{1.3.18}$$

This is already known to be true for the term G_0 appearing in G. Let us denote the second term in eq. (1.3.12) by F,

$$F(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \frac{1}{\Delta(\lambda)} \int D(\boldsymbol{x}, \boldsymbol{z}; \lambda)\, G_0(\boldsymbol{z}, \boldsymbol{y}; \lambda)\, d\boldsymbol{z}. \tag{1.3.19}$$

Then it is obvious that F belongs to $\mathfrak{L}^2$, since D belongs to $\mathfrak{L}^2$ and G_0 represents a bounded operator. Therefore,

$$\left.\begin{aligned} &\iint |\, \bar{g}(\boldsymbol{x})\, F(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, f(\boldsymbol{y})\, |\, d\boldsymbol{x}\, d\boldsymbol{y} \\ &\leq \left[\iint |\, F(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, |^2 d\boldsymbol{x}\, d\boldsymbol{y} \int |\, g(\boldsymbol{x}')\, |^2 d\boldsymbol{x}' \int |\, f(\boldsymbol{y}')\, |^2 d\boldsymbol{y}' \right]^{\frac{1}{2}} < \infty, \end{aligned}\right\} \tag{1.3.20}$$

so that by Fubini's theorem

$$\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int F(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, f(\boldsymbol{y})\, d\boldsymbol{y} = \int f(\boldsymbol{y})\, d\boldsymbol{y} \int F(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{1.3.21}$$

But from eq. (1.3.13) and eqs. (1.3.16) and (1.3.17) we have

$$\left.\begin{aligned} \int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int F(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, f(\boldsymbol{y})\, d\boldsymbol{y} &= \int f(\boldsymbol{y})\, d\boldsymbol{y} \int \overline{F}(\boldsymbol{y}, \boldsymbol{x}; \bar{\lambda})\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \\ &= \int f(\boldsymbol{y})\, d\boldsymbol{y} \int F(\boldsymbol{y}, \boldsymbol{x}; \lambda)\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x}. \end{aligned}\right\} \tag{1.3.22}$$

In view of the arbitrariness of f and g in $\mathfrak{L}^2$ and the fact that F belongs to $\mathfrak{L}^2$, this implies that

$$F(\boldsymbol{x}, \boldsymbol{y}; \lambda) = F(\boldsymbol{y}, \boldsymbol{x}; \lambda), \tag{1.3.23}$$

and from this we get eq. (1.3.18). In this argument use is made of the known interchangeability of the integrals to establish the symmetry. Now that we have the symmetry of the Green function, it follows, as in eq. (1.3.14), that

$$\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int G(\boldsymbol{x}, \boldsymbol{y}; \lambda) f(\boldsymbol{y})\, d\boldsymbol{y} = \int f(\boldsymbol{y})\, d\boldsymbol{y} \int G(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, \bar{g}(\boldsymbol{x})\, d\boldsymbol{x}. \tag{1.3.24}$$

According to the resolvent equation (1.2.14), the Green function satisfies

$$\int G(\boldsymbol{x}, \boldsymbol{z}; \lambda) f(\boldsymbol{z})\, d\boldsymbol{z} = \int G_0(\boldsymbol{x}, \boldsymbol{z}; \lambda) f(\boldsymbol{z})\, d\boldsymbol{z} - \int G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda) V(\boldsymbol{y})\, d\boldsymbol{y} \int G(\boldsymbol{y}, \boldsymbol{z}; \lambda) f(\boldsymbol{z})\, d\boldsymbol{z}. \tag{1.3.25}$$

If it is now recalled that $G_0 V$ belongs to $\mathfrak{L}^2$, and if eq. (1.3.24) is taken into account, the argument used in the previous section to get eq. (1.3.11) shows that in eq. (1.3.25) the integrations with respect to $\boldsymbol{y}$ and $\boldsymbol{z}$ may be interchanged. This yields the kernel

$$-\int G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda) V(\boldsymbol{y})\, G(\boldsymbol{y}, \boldsymbol{z}; \lambda)\, d\boldsymbol{y}, \tag{1.3.26}$$

which obviously belongs to $\mathfrak{L}^2$. Likewise, the second term on the right-hand side of eq. (1.3.12) belongs to $\mathfrak{L}^2$. Since both kernels give the same result when multiplied by $f(\boldsymbol{z})$ and integrated over $\boldsymbol{z}$, the two kernels must be equal for almost every $\boldsymbol{x}$ and $\boldsymbol{z}$. In other words, along with eq. (1.3.12) we have

$$G(\boldsymbol{x}, \boldsymbol{z}; \lambda) = G_0(\boldsymbol{x}, \boldsymbol{z}; \lambda) - \int G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda) V(\boldsymbol{y})\, G(\boldsymbol{y}, \boldsymbol{z}; \lambda)\, d\boldsymbol{y}. \tag{1.3.27}$$

1.3.3. Analytic properties

We now pass on to a study of the analytic properties of $D(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ and $\Delta(\lambda)$ considered as functions of λ. To this end, we first observe that, according to eqs. (1.2.11) and (1.2.12),

$$\left.\begin{aligned} \|R(\lambda) f\|^2 &= (R(\lambda) f, R(\lambda) f) = (f, R(\bar{\lambda}) R(\lambda) f) \\ &= \frac{1}{\lambda - \bar{\lambda}} [(f, R(\lambda) f) - (f, R(\bar{\lambda}) f)] \leq \frac{1}{|\operatorname{Im} \lambda|} \|f\|\, \|R(\lambda) f\|. \end{aligned}\right\} \tag{1.3.28}$$

Denoting the norm of $R(\lambda)$ by $\|R(\lambda)\|$, we thus obtain

$$\|R(\lambda)\| \leq \frac{1}{|\operatorname{Im} \lambda|}. \tag{1.3.29}$$

This is a useful bound for non-real λ. However, if there is no interaction the resolvent does not only exist off the real axis, it also exists, and is bounded, for negative real λ. As a matter of fact, in the case of two particles we have

$$\left.\begin{aligned}\int d\boldsymbol{x}\left|\int \frac{e^{i\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|} f(\boldsymbol{y})\, d\boldsymbol{y}\right|^2 &\le \iiint \frac{e^{-\operatorname{Im}\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{y}|}}{4\pi|\boldsymbol{x}-\boldsymbol{y}|}\,\frac{e^{-\operatorname{Im}\sqrt{\lambda}|\boldsymbol{x}-\boldsymbol{z}|}}{4\pi|\boldsymbol{x}-\boldsymbol{z}|}\,|f(\boldsymbol{y})|\,|f(\boldsymbol{z})|\,d\boldsymbol{x}\,d\boldsymbol{y}\,d\boldsymbol{z}\\ &= \iiint \frac{e^{-\operatorname{Im}\sqrt{\lambda}|\boldsymbol{y}'|}}{4\pi|\boldsymbol{y}'|}\,\frac{e^{-\operatorname{Im}\sqrt{\lambda}|\boldsymbol{z}'|}}{4\pi|\boldsymbol{z}'|}\,|f(\boldsymbol{x}-\boldsymbol{y}')|\,|f(\boldsymbol{x}-\boldsymbol{z}')|\,d\boldsymbol{x}\,d\boldsymbol{y}'\,d\boldsymbol{z}'\\ &\le \left[\int \frac{e^{-\operatorname{Im}\sqrt{\lambda}|\boldsymbol{y}|}}{4\pi|\boldsymbol{y}|}\,d\boldsymbol{y}\right]^2 \|f\|^2 = \frac{1}{(\operatorname{Im}\sqrt{\lambda})^4}\,\|f\|^2,\end{aligned}\right\} \tag{1.3.30}$$

so that here

$$\|R_0(\lambda)\| \le \frac{1}{(\operatorname{Im}\sqrt{\lambda})^2}\,. \tag{1.3.31}$$

As we shall see later, this inequality applies also to larger numbers of particles.

Another relation which follows directly from eq. (1.2.11) is

$$\frac{d}{d\lambda}(g, R(\lambda) f) = (g, R(\lambda) R(\lambda) f). \tag{1.3.32}$$

Extending this to higher derivatives yields the power series

$$(g, R(\lambda) f) = \sum_{p=0}^{\infty} (\lambda-\mu)^p (g, [R(\mu)]^{p+1} f), \tag{1.3.33}$$

which converges provided

$$|\lambda-\mu|\,\|R(\mu)\| < 1, \tag{1.3.34}$$

and so certainly when λ is inside the circle with centre μ which is tangent to the real axis, by eq. (1.3.29). Clearly the left-hand side of eq. (1.3.33) is regular inside this circle, and by letting μ take all non-real values, it follows that it is in fact regular for all non-real λ. With the additional information supplied by the bound (1.3.31) for $\|R_0(\lambda)\|$, it follows that $(g, R_0(\lambda) f)$ is even regular in a larger region, viz. inside the λ-plane cut along the real axis from 0 to ∞.

For the purpose of our investigation of the functions $D(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ and $\Delta(\lambda)$, the foregoing is now extended to functions of the form $(g, VR_0(\lambda) f)$. The point is here that in general Vg does not belong to $\mathfrak{L}^2$. However, let us choose a sequence g_N in $\mathfrak{L}^2$ such that Vg_N belongs to $\mathfrak{L}^2$ for every N, while g_N tends to g in mean square,

$$\lim_{N\to\infty} \|g_N - g\| = 0. \tag{1.3.35}$$

Such a sequence certainly exists. For since $\mathfrak{D}(H_0)$ is dense in $\mathfrak{L}^2$, every g in $\mathfrak{L}^2$ can be approximated in mean square by a sequence in $\mathfrak{D}(H_0)$. And if g_N is in $\mathfrak{D}(H_0)$, the function Vg_N belongs to $\mathfrak{L}^2$, by eq. (1.2.9).

For every N, the result (1.3.33) yields the expansion

$$(Vg_N, R_0(\lambda) f) = \sum_{p=0}^{\infty} (\lambda-\mu)^p (Vg_N, [R_0(\mu)]^{p+1} f). \tag{1.3.36}$$

Use can now be made of the relation

$$(Vg_N, R_0(\lambda)f) = (g_N, VR_0(\lambda)f) \tag{1.3.37}$$

and of the fact that $VR_0(\lambda)$ is a bounded operator, again by eq. (1.2.9). Since $VR_0(\lambda)$ is bounded, it follows from eq. (1.3.35) that

$$\lim_{N\to\infty}(g_N, VR_0(\lambda)f) = (g, VR_0(\lambda)f). \tag{1.3.38}$$

Hence, both members of eq. (1.3.37) tend to a limit as N tends to ∞, and so does each term on the right-hand side of eq. (1.3.36). Also, the convergence of the series in eq. (1.3.36) is uniform with respect to N whenever λ satisfies eq. (1.3.34). Then the sum of the limits equals the limit of the sum. Summarizing, we thus get the result

$$(g, VR_0(\lambda)f) = \sum_{p=0}^{\infty}(\lambda-\mu)^p (g, V[R_0(\mu)]^{p+1}f), \tag{1.3.39}$$

which shows that $(g, VR_0(\lambda)f)$ is regular in the same region as $(g, R_0(\lambda)f)$.

The argument is now easily extended to the case in which g itself is a function of λ, and is of the form $R_0(\bar{\lambda})h$ or $R_0(\bar{\lambda})Vh$. By the previous result and eq. (1.2.12), each term of the series in eq. (1.3.39) is still regular, though more complicated than with λ-independent g. The series converges uniformly with respect to λ in any closed region in which λ satisfies eq. (1.3.34). From this it follows as before that the sum is regular in the λ-plane cut from 0 to ∞. The same applies to general expressions of the form

$$(g, R_0(\lambda)[VR_0(\lambda)]^q f),\ (g, [VR_0(\lambda)]^q f)\ (q = 0, 1, 2, \ldots). \tag{1.3.40}$$

Now by writing out the quantity

$$\int \bar{g}(\boldsymbol{x})\,d\boldsymbol{x}\int D_p(\boldsymbol{x},\boldsymbol{y};\lambda)\,d\boldsymbol{y}\int G_0(\boldsymbol{y},\boldsymbol{z};\lambda)f(\boldsymbol{z})\,d\boldsymbol{z} \tag{1.3.41}$$

with the help of eq. (1.3.6), it can be reduced to functions $\Delta_q(\lambda)$ and expressions of the first form (1.3.40). The latter being regular in a certain region of the λ-plane, we now turn to the functions $\Delta_q(\lambda)$. Let us take in particular

$$\Delta_2(\lambda) = -\frac{1}{2}\iint [G_0(\boldsymbol{x},\boldsymbol{y};\lambda)]^2 V(\boldsymbol{x})V(\boldsymbol{y})\,d\boldsymbol{x}\,d\boldsymbol{y}. \tag{1.3.42}$$

Here we require the fact that the kernel G_0V belongs to $\mathfrak{L}^2$. By virtue of this, it can be developed in mean square in terms of a complete orthonormal set $f_q(\boldsymbol{x})$ $(q = 1, 2, \ldots)$ in $\mathfrak{L}^2$, according to

$$\left.\begin{aligned} -G_0(\boldsymbol{x},\boldsymbol{y};\lambda)V(\boldsymbol{y}) &= \sum_{q,r=1}^{\infty} K_{qr}(\lambda) f_q(\boldsymbol{x})\bar{f}_r(\boldsymbol{y}), \\ K_{qr}(\lambda) &= -\iint \bar{f}_q(\boldsymbol{x})G_0(\boldsymbol{x},\boldsymbol{y};\lambda)V(\boldsymbol{y})f_r(\boldsymbol{y})\,d\boldsymbol{x}\,d\boldsymbol{y}. \end{aligned}\right\} \tag{1.3.43}$$

In terms of the coefficients K_{qr}, the $\mathfrak{L}^2$-norm of $G_0 V$ equals

$$|G_0(\lambda)V| = \left[\sum_{q,r=1}^{\infty} |K_{qr}(\lambda)|^2\right]^{\frac{1}{2}}. \tag{1.3.44}$$

The function Δ_2 takes the form

$$\Delta_2(\lambda) = -\frac{1}{2}\sum_{q,r=1}^{\infty} K_{qr}(\lambda) K_{rq}(\lambda). \tag{1.3.45}$$

Equation (1.3.44) implies the convergence of the series on the right-hand side. In view of this, Schwarz's inequality shows that the series for $\Delta_2(\lambda)$ is absolutely convergent, its partial sums not exceeding $|G_0(\lambda)V|^2/2$.

If we now consider in particular

$$\Delta_2^N(\lambda) = -\frac{1}{2}\sum_{q,r=1}^{N} K_{qr}(\lambda) K_{rq}(\lambda), \tag{1.3.46}$$

we know from the foregoing that it is regular in the λ-plane cut along the real axis from 0 to ∞, for the sum is confined to a finite number of terms, and each term is regular by our previous results. In other words, each function of the sequence $\Delta_2^N(\lambda)$ $(N = 1, 2, \ldots)$ is regular. The sequence tends to $\Delta_2(\lambda)$ as N tends to ∞, and it is bounded uniformly in N in any region in which $|G_0(\lambda)V|$ is bounded, i.e. in any region in which $\operatorname{Im}\sqrt{\lambda} \geq \varepsilon > 0$ (cf. eq. (1.3.4)). From this it follows by Vitali's theorem (TITCHMARSH(15) section 5. 21) that $\Delta_2(\lambda)$ is regular in the region $\operatorname{Im}\sqrt{\lambda} > 0$.

Combining various arguments presented in the foregoing we now easily see that each function $\Delta_p(\lambda)$ is regular in the region $\operatorname{Im}\sqrt{\lambda} > 0$, and that the same applies to the functions

$$\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int f(\boldsymbol{z})\, d\boldsymbol{z} \int D_p(\boldsymbol{x}, \boldsymbol{y}; \lambda)\, G_0(\boldsymbol{y}, \boldsymbol{z}; \lambda)\, d\boldsymbol{y}. \tag{1.3.47}$$

Taking into account the convergence properties of the series for $\Delta(\lambda)$ implied by eq. (1.3.9), a second application of Vitali's theorem yields the result that $\Delta(\lambda)$ is regular in the region $\operatorname{Im}\sqrt{\lambda} > 0$. The same applies to the sum with respect to p of all expressions (1.3.47). If we now recall the expression (1.3.12) for the Green function, we see that

$$(g, R(\lambda) f) = \int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int G(\boldsymbol{x}, \boldsymbol{z}; \lambda) f(\boldsymbol{z})\, d\boldsymbol{z} \tag{1.3.48}$$

is regular in any region with $\operatorname{Im}\sqrt{\lambda} > 0$ where $\Delta(\lambda)$ does not vanish.

Obviously, the interest of this conclusion does not derive so much from what it says about the analytic behaviour of the resolvent off the real axis. This already follows from general principles (see the discussion following eq. (1.3.33)). The significant result is the information on the properties of the resolvent in the neighbourhood of

the negative real axis. This is by no means trivial, as it depends on the nature of the interaction. Whereas in general the resolvent of an arbitrary self-adjoint operator exists only for non-real λ, we see here that in the case of two particles with square-integrable interaction it can be continued analytically across the negative real axis. As for the restriction that $\Delta(\lambda)$ must not vanish, since $\Delta(\lambda)$ is regular, its zeros are isolated points. Hence they give rise only to poles of the resolvent. It is these poles to which we now direct our attention.

1.3.4. Bound states

It was remarked above that, if $\Delta(\lambda)$ has zeros, these give rise to poles of the resolvent. However, we saw from eqs. (1.3.29) and (1.3.33) that the resolvent is regular off the real axis, so that the possible zeros of $\Delta(\lambda)$ must be confined to the real axis. In view of the regularity of $\Delta(\lambda)$, they cannot have an accumulation point in the region $\operatorname{Im}\sqrt{\lambda} \geq \varepsilon > 0$, except perhaps the point at infinity. The latter possibility is easily ruled out, however, by the following argument. According to the resolvent equation (1.2.14), we formally have

$$(g, R(\lambda)f) - (g, R_0(\lambda)f) = \sum_{p=1}^{\infty} (-1)^p (g, [R_0(\lambda)V]^p R_0(\lambda)f). \tag{1.3.49}$$

In general this series is not convergent. But if $\operatorname{Im}\sqrt{\lambda}$ is sufficiently large, it follows from eq. (1.3.4) that the $\mathfrak{L}^2$-norm of $R_0 V$ is sufficiently small,

$$|R_0(\lambda)V| < 1 \quad \left(\operatorname{Im}\sqrt{\lambda} > \frac{1}{8\pi}\int [V(\boldsymbol{x})]^2 d\boldsymbol{x}\right). \tag{1.3.50}$$

Under this condition, the series on the right-hand side of eq. (1.3.49) is easily shown to converge and to equal the left-hand side. It thus exhibits the resolvent as a bounded operator, also on the negative real axis. There can therefore be no poles in the neighbourhood of the point at infinity, which is what we wanted to show. Summarizing, we see that on that part of the negative real axis on which $\operatorname{Im}\sqrt{\lambda} \geq \varepsilon > 0$, the function $\Delta(\lambda)$ may have a finite number of zeros at most.

Let us now choose a particular zero $\lambda = \lambda_\alpha$, and let us assume that it is of order q. Then it follows from eq. (1.3.11) that, in the neighbourhood of λ_α, the quantity $(g, R(\lambda)f)$ can be expanded in a Laurent series the most singular term of which is proportional to $(\lambda - \lambda_\alpha)^{-q}$. Hence, in the expansion of $(d/d\lambda)(g, R(\lambda)f)$, the most singular term is proportional to $(\lambda - \lambda_\alpha)^{-q-1}$. But, according to eq. (1.3.32), it is also proportional to $(\lambda - \lambda_\alpha)^{-2q}$, so that $q = 1$. The resolvent can therefore have only simple poles.

At this stage it is convenient to consider the function

$$P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) = \lim_{\lambda \to \lambda_\alpha} \frac{\lambda - \lambda_\alpha}{\Delta(\lambda)} \int D(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha)\, G_0(\boldsymbol{y}, \boldsymbol{z}; \lambda_\alpha)\, d\boldsymbol{y}, \tag{1.3.51}$$

which, according to eq. (1.3.12), is the "residue" of $G(\boldsymbol{x}, \boldsymbol{z}; \lambda)$ at $\lambda = \lambda_\alpha$. From the symmetry relations for the Green function, eqs. (1.3.17) and (1.3.18), it follows that P is real, and that it satisfies

$$P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) = P(\boldsymbol{z}, \boldsymbol{x}; \lambda_\alpha). \tag{1.3.52}$$

Furthermore, P is a Hilbert-Schmidt kernel. By virtue of all this, P can be developed in mean square in a series of orthonormal eigenfunctions in $\mathfrak{L}^2$ according to (RIESZ and SZ.-NAGY(3) section 97)

$$P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) = \sum_{q=1}^{\infty} \mu_{\alpha q} \varphi_{\alpha q}(\boldsymbol{x}) \bar{\varphi}_{\alpha q}(\boldsymbol{z}), \tag{1.3.53}$$

$$|P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha)| = \left[\sum_{q=1}^{\infty}{}' \mu_{\alpha q}^2\right]^{\frac{1}{2}}. \tag{1.3.54}$$

Here the eigenvalues μ are real, since P is hermitian. Hence, if φ is an eigenfunction corresponding to a certain eigenvalue μ, so is $\bar{\varphi}$. The above series can therefore be arranged in such a way that the functions $\varphi_{\alpha q}$ are all real. This will always be implied in the following, unless otherwise stated.

It is clear from eq. (1.3.12) that, if $(g, R(\lambda) f)$ is expanded in a Laurent series in powers of $\lambda - \lambda_\alpha$, its principal part takes the form

$$\frac{1}{\lambda - \lambda_\alpha} \int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) f(\boldsymbol{z})\, d\boldsymbol{z}. \tag{1.3.55}$$

Hence, equating the most singular terms in the expansions of $(d/d\lambda)\,(g, R(\lambda) f)$ and $(g, R(\lambda)\, R(\lambda) f)$ yields

$$\left.\begin{aligned} &\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) f(\boldsymbol{z})\, d\boldsymbol{z} \\ &= -\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int P(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha)\, d\boldsymbol{y} \int P(\boldsymbol{y}, \boldsymbol{z}; \lambda_\alpha) f(\boldsymbol{z})\, d\boldsymbol{z}. \end{aligned}\right\} \tag{1.3.56}$$

In view of the arbitrariness of f and g in $\mathfrak{L}^2$, this shows that $-P$ represents a projection. As a result the eigenvalues all satisfy $\mu_{\alpha q} = -1$, the number of independent eigenfunctions is finite by eq. (1.3.54), and eq. (1.3.53) holds for almost every $\boldsymbol{x}$ and $\boldsymbol{z}$.

To get more insight into the properties of the eigenfunctions φ, we consider once more the resolvent equation (1.2.14),

$$R(\lambda) f = R_0(\lambda) f - R_0(\lambda) V R(\lambda) f. \tag{1.3.57}$$

Substituting the Green function on both sides, developing in a Laurent series in powers of $\lambda - \lambda_\alpha$, and equating principal parts yields

$$\int P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) f(\boldsymbol{z})\, d\boldsymbol{z} = -\int G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha) V(\boldsymbol{y})\, d\boldsymbol{y} \int P(\boldsymbol{y}, \boldsymbol{z}; \lambda_\alpha) f(\boldsymbol{z})\, d\boldsymbol{z}. \tag{1.3.58}$$

Taking in particular $f = \varphi_{\alpha q}$ gives

$$\varphi_{\alpha q}(\boldsymbol{x}) = -\int G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha) V(\boldsymbol{y}) \varphi_{\alpha q}(\boldsymbol{y})\, d\boldsymbol{y}. \tag{1.3.59}$$

We now want to show that $V\varphi_{\alpha q}$ belongs to $\mathfrak{L}^2$. To do this, we recall first of all that, since $D(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha)$ belongs to $\mathfrak{L}^2$, it is a square-integrable function of $\boldsymbol{y}$ for almost every $\boldsymbol{x}$, by Fubini's theorem. Further, $VR_0(\lambda_\alpha)$ is a bounded operator. Denoting its bound by $\| VR_0(\lambda_\alpha) \|$, we deduce from eq. (1.3.51) that

$$\int [V(\boldsymbol{z}) P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha)]^2 d\boldsymbol{z} \leq \text{const.} \| VR_0(\lambda_\alpha) \|^2 \int | D(\boldsymbol{x}, \boldsymbol{y}; \lambda_\alpha) |^2 d\boldsymbol{y}, \tag{1.3.60}$$

$$\iint [V(\boldsymbol{z}) P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha)]^2 d\boldsymbol{z}\, d\boldsymbol{x} \leq \text{const.} \| VR_0(\lambda_\alpha) \|^2 | D(\lambda_\alpha) |^2. \tag{1.3.61}$$

From this it follows with Schwarz's inequality that

$$\left.\begin{aligned} \int | V(\boldsymbol{z}) \varphi_{\alpha q}(\boldsymbol{z}) |^2 d\boldsymbol{z} &= \int | V(\boldsymbol{z}) \int P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha) \varphi_{\alpha q}(\boldsymbol{x})\, d\boldsymbol{x} |^2 d\boldsymbol{z} \\ &\leq \iint [V(\boldsymbol{z}) P(\boldsymbol{x}, \boldsymbol{z}; \lambda_\alpha)]^2 d\boldsymbol{z}\, d\boldsymbol{x} \int | \varphi_{\alpha q}(\boldsymbol{y}) |^2 d\boldsymbol{y} < \infty, \end{aligned}\right\} \tag{1.3.62}$$

which is what we wanted to show.

It is now permitted to apply the operator $H_0 - \lambda_\alpha$ to both sides of eq. (1.3.59). This yields

$$\left.\begin{aligned} (H_0 - \lambda_\alpha) \varphi_{\alpha q}(\boldsymbol{x}) &= -V(\boldsymbol{x}) \varphi_{\alpha q}(\boldsymbol{x}), \\ (H - \lambda_\alpha) \varphi_{\alpha q}(\boldsymbol{x}) &= 0. \end{aligned}\right\} \tag{1.3.63}$$

In other words, φ satisfies the Schrödinger equation. Since it belongs to $\mathfrak{L}^2$, it is the eigenfunction for a bound state, with energy λ_α.

Since in the region with $\operatorname{Im} \sqrt{\lambda} \geq \varepsilon > 0$ the number of poles of the resolvent is finite, the above argument can easily be extended to show that the Green function can be written in the form

$$G(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \sum_{\substack{\alpha=1 \\ \operatorname{Im}\sqrt{\lambda_\alpha} \geq \varepsilon}}^{N(\varepsilon)} \sum_{q_\alpha=1}^{Q_\alpha} \frac{\varphi_{\alpha q_\alpha}(\boldsymbol{x})\, \bar{\varphi}_{\alpha q_\alpha}(\boldsymbol{y})}{\lambda_\alpha - \lambda} + G_B(\boldsymbol{x}, \boldsymbol{y}; \lambda; \varepsilon), \tag{1.3.64}$$

where G_B is determined by the requirement that for $\operatorname{Im} \sqrt{\lambda} \geq \varepsilon > 0$ it be the kernel of a bounded operator. In eq. (1.3.64) Q_α and $N(\varepsilon)$ are certain finite integers. This equation is the analogue of the well-known heuristic expansion in terms of a complete orthonormal set ψ_q $(q = 1, 2, \ldots)$,

$$G(\boldsymbol{x}, \boldsymbol{y}; \lambda) = \sum_{q=1}^{\infty} \frac{\psi_q(\boldsymbol{x}) \bar{\psi}_q(\boldsymbol{y})}{\lambda_q - \lambda}, \tag{1.3.65}$$

which in general can be justified only when the spectrum is purely discrete.

The formalism developed so far does not tell anything about the behaviour of the resolvent in the neighbourhood of the positive real axis. When we pass on to the theory of scattering in a forthcoming paper, we shall see that there is a continuous spectrum from 0 to ∞. It will be discussed how this is related to scattering cross-sections. As regards the interval on the negative real axis where $0 \leq \operatorname{Im} \sqrt{\bar{\lambda}} < \varepsilon$, we should like to know if the number of poles on this is finite or infinite. It is obvious that in the latter case the origin would be the only point of accumulation. It was shown by BARGMANN(16) that, in the case of two particles with a spherically symmetric interaction satisfying

$$\int_{-\infty}^{\infty} \frac{|V(|\boldsymbol{x}|)|}{|\boldsymbol{x}|} d^3\boldsymbol{x} < \infty, \tag{1.3.66}$$

the number of poles is finite. However, Bargmann's method is restricted to separable Hamiltonians, as it is based on the theory of ordinary differential equations. We have not succeeded in recasting his ideas so as to make them applicable to larger numbers of particles. It is for this reason that the behaviour of the resolvent in the neighbourhood of the origin is left out of consideration in the present paper.

1.4. Spectral theory

1.4.1. The integral representation of the resolvent

Before we can pass on to systems of three or more particles, we must give a short summary of some elements of the spectral theory of self-adjoined operators. Most of the material of the present section can be obtained from ACHIESER and GLASMANN(2) or other books on Hilbert space. However, with a view to future applications, we have preferred not to stress the well-known Stieltjes-integral approach to the resolution of the identity. Instead of this a formalism is given mainly in terms of limits of Riemann integrals of resolvents off the real axis. In future sections this will be found to be very useful, as it bridges the gap between the resolvent off the real axis, which is a fairly manageable quantity, and the spectral properties of the Hamiltonian, which have to do with the much more tricky properties of the resolvent in the neighbourhood of the real axis. Our final formulas are essentially the ones given by TITCHMARSH(8) ch. XII, and the present paper owes very much to his book.

Let us start by considering $(f, R(\lambda)f)$ as a function of λ, for fixed f in $\mathfrak{Q}^2$. It was discussed in section 1.3.3 that this function is regular off the real axis. It follows from eqs. (1.2.11) and (1.2.12) that

$$\frac{1}{\operatorname{Im}\lambda} \operatorname{Im}(f, R(\lambda) f) = \frac{1}{\lambda - \bar{\lambda}} (f, [R(\lambda) - R(\bar{\lambda})] f) = (f, R(\bar{\lambda}) R(\lambda) f) \geq 0, \tag{1.4.1}$$

so that in the upper half-plane the function $(f, R(\lambda)f)$ has a non-negative imaginary part. Furthermore, it satisfies

$$|(f, R(\lambda) f)| \leq \frac{1}{|\operatorname{Im} \lambda|} \|f\|^2 \tag{1.4.2}$$

by eq. (1.3.29). From all this it follows that there exists a non-decreasing function of bounded variation $E(l; f, f)$ such that (ACHIESER and GLASMANN(2) section 59)

$$(f, R(\lambda) f) = \int_{-\infty}^{\infty} \frac{dE(l; f, f)}{l - \lambda} \quad (\operatorname{Im} \lambda > 0), \tag{1.4.3}$$

$$\frac{1}{2} E(l-0; f, f) + \frac{1}{2} E(l+0; f, f) = \text{const.} + \lim_{\varepsilon \to 0} \frac{1}{2\pi i} \int_0^l (f, [R(m+i\varepsilon) - R(m-i\varepsilon)] f)\, dm. \tag{1.4.4}$$

If, in addition to satisfying eq. (1.4.4), $E(l; f, f)$ is chosen in such a way that

$$\left.\begin{aligned} E(-\infty; f, f) &\equiv \lim_{l \to -\infty} E(l; f, f) = 0, \\ E(l-0; f, f) &= E(l; f, f), \end{aligned}\right\} \tag{1.4.5}$$

it is uniquely determined.

If we want to consider a general expression of the form $(g, R(\lambda) f)$, it is useful to split this into four terms according to

$$\left.\begin{aligned} 4(g, R(\lambda) f) = (g+f, R(\lambda)[g+f]) - (g-f, R(\lambda)[g-f]) \\ - i(g+if, R(\lambda)[g+if]) + i(g-if, R(\lambda)[g-if]). \end{aligned}\right\} \tag{1.4.6}$$

Here each term allows an integral representation of the form (1.4.3), and so the same holds true for the sum. In other words, there is a relation

$$(g, R(\lambda) f) = \int_{-\infty}^{\infty} \frac{dE(l; g, f)}{l - \lambda} \quad (\operatorname{Im} \lambda > 0), \tag{1.4.7}$$

$E(l; g, f)$ satisfying equations of the forms (1.4.4) and (1.4.5). The function $E(l; g, f)$ is no longer non-decreasing, but since it is a linear combination of non-decreasing functions, it is still of bounded variation.

It follows from eq. (1.4.3) that

$$\left.\begin{aligned} \frac{1}{2\pi i} \int_{-\infty}^{\infty} (f, [R(m+i\varepsilon) - R(m-i\varepsilon)] f)\, dm &= \frac{1}{\pi} \int_{-\infty}^{\infty} dm \int_{-\infty}^{\infty} \frac{\varepsilon}{(m-l)^2 + \varepsilon^2}\, dE(l; f, f) \\ = \frac{1}{\pi} \int_{-\infty}^{\infty} dE(l; f, f) \int_{-\infty}^{\infty} \frac{\varepsilon}{(m-l)^2 + \varepsilon^2}\, dm &= E(\infty; f, f), \end{aligned}\right\} \tag{1.4.8}$$

where the inversion of the integrations is justified by Fubini's theorem for Stieltjes integrals (SAKS(17) ch. III, section 8; also WIDDER(18) ch. I, section 15). If the way

is remembered in which $(g, R(\lambda)f)$ was written as a sum of four terms, eq. (1.4.6), a similar relation is obtained for $E(\infty; g, f)$. Moreover,

$$\frac{2}{\pi}\int_{-\infty}^{\infty} |(g, [R(m+i\varepsilon) - R(m-i\varepsilon)]f)|\, dm$$

$$\leq E(\infty; g+f, g+f) + E(\infty; g-f, g-f) + E(\infty; g+if, g+if) + E(\infty; g-if, g-if), \quad (1.4.9)$$

uniformly in ε. From this it is obvious that as a function of l the integral

$$\frac{1}{2\pi i}\int_0^l (g, [R(m+i\varepsilon) - R(m-i\varepsilon)]f)\, dm \quad (1.4.10)$$

is of bounded variation uniformly with respect to ε. As ε tends to 0, this integral tends to const. $+ [E(l-0; g, f) + E(l+0; g, f)]/2$ (cf. eq. (1.4.4)). Also, $(l-\lambda)^{-1}$ is a continuous function of l. Hence, according to the Helly-Bray theorem on limits of Stieltjes integrals (WIDDER (18) ch. I, theorem 16.4), we have

$$(g, R(\lambda)f) = \lim_{\substack{L\to-\infty \\ T\to\infty}} \int_L^T \frac{dE(l; g, f)}{l-\lambda} = \lim_{\substack{L\to-\infty \\ T\to\infty}} \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_L^T (g, [R(l+i\varepsilon) - R(l-i\varepsilon)]f)\frac{dl}{l-\lambda}. \quad (1.4.11)$$

In this formula it is in general necessary to perform the limit with respect to ε first, while the integration interval (L, T) is still finite, because the validity of the Helly-Bray theorem is restricted to finite intervals.

1.4.2. Spectral resolutions

To get more insight into the quantity $E(\infty; g, f)$, we now consider the relation

$$\lambda(g, R(\lambda)f) = -(g, f) + (g, HR(\lambda)f). \quad (1.4.12)$$

With Minkowski's inequality, this yields a bound for the operator $HR(\lambda)$ according to

$$\|HR(\lambda)f\| \leq \|f\| + |\lambda|\,\|R(\lambda)f\| \leq \left[1 + \frac{|\lambda|}{|\mathrm{Im}\,\lambda|}\right]\|f\|. \quad (1.4.13)$$

Now, if g belongs to $\mathfrak{D}(H)$, we clearly have

$$\lim_{\nu\to\infty} (g, HR(i\nu)f) = 0. \quad (1.4.14)$$

But in view of eq. (1.4.13), this must also hold true for any arbitrary g in $\mathfrak{L}^2$. For if we have any g in $\mathfrak{L}^2$, we can always choose a sequence g_N in $\mathfrak{D}(H)$ which tends to g in mean square. In terms of this we get

$$\left.\begin{aligned} &|(g, HR(i\nu)f)| = |(g-g_N, HR(i\nu)f)+(g_N, HR(i\nu)f)| \\ &\leq \|g-g_N\|\,\|HR(i\nu)f\|+|(g_N, HR(i\nu)f)| \leq 2\,\|g-g_N\|\,\|f\|+\|Hg_N\|\,\|R(i\nu)f\|, \end{aligned}\right\} \quad (1.4.15)$$

and this can be made arbitrarily small by choosing first N, next ν sufficiently large.

With eq. (1.4.12) it now follows that

$$\lim_{\nu\to\infty} i\nu(g, R(i\nu)f) = -(g, f). \quad (1.4.16)$$

Combining this with eq. (1.4.7) yields

$$(g, f) = -\lim_{\nu\to\infty} i\nu\int_{-\infty}^{\infty} \frac{dE(l; g, f)}{l-i\nu} = \int_{-\infty}^{\infty} dE(l; g, f) = E(\infty; g, f), \quad (1.4.17)$$

where use is made of the normalization $E(-\infty; g, f) = 0$ (cf. eq. (1.4.5)). By eq. (1.4.4), an alternative way of writing this is

$$(g, f) = \lim_{\substack{L\to-\infty \\ T\to\infty}} \lim_{\varepsilon\to 0} \frac{1}{2\pi i}\int_L^T (g, [R(l+i\varepsilon)-R(l-i\varepsilon)]f)\,dl. \quad (1.4.18)$$

In this form the expression is a generalization of the well-known expansion of (g, f) in terms of a complete orthonormal set of functions in $\mathfrak{L}^2$. For let us consider the simple case that the spectrum is discrete, and such that the resolvent can be represented by a Green function of the form (1.3.65). Under these circumstances we have

$$\left.\begin{aligned} (g, [R(l+i\varepsilon)-R(l-i\varepsilon)]f) &= \sum_{q=1}^{\infty} \bar{g}_q f_q \frac{2i\varepsilon}{(\lambda_q - l)^2+\varepsilon^2}, \\ f_q &= \int \bar{\psi}_q(\boldsymbol{y}) f(\boldsymbol{y})\,d\boldsymbol{y}, \end{aligned}\right\} \quad (1.4.19)$$

so that the expansion (1.4.18) takes the well-known form

$$(g, f) = \sum_{q=1}^{\infty} \bar{g}_q f_q. \quad (1.4.20)$$

We note here without proof that the function $E(l; g, f)$ satisfies an equation of the form

$$E(l; g, f) = (g, E(l)f), \quad (1.4.21)$$

where $E(l)$ is a projection operator with the properties (Achieser and Glasmann(2) section 65)

$$\left.\begin{aligned} &E(l)\,E(m) = E(\min(l, m)), \\ &E(-\infty) = 0, \quad E(\infty) = 1, \\ &E(l-0) = E(l). \end{aligned}\right\} \quad (1.4.22)$$

The operator $E(l)$ is called the resolution of the identity. Some authors normalize $E(l)$ in such a way that $E(l+0) = E(l)$ (see STONE(4) definition 5.1). This will give no confusion in the following. If there is a discrete spectrum with a Green function of the form (1.3.65), we have in any case

$$\left.\begin{aligned}(g,[E(\lambda_p+0)-E(\lambda_p-0)]f) &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi}\int_{\lambda_p-\delta}^{\lambda_p+\delta}\left[\sum_{q=1}^{\infty}{}'\bar{g}_q f_q\frac{\varepsilon}{(\lambda_q-l)^2+\varepsilon^2}\right]dl\\ &= \lim_{\delta\to 0}\lim_{\varepsilon\to 0}\frac{1}{\pi}\sum_{q=1}^{\infty}{}'\bar{g}_q f_q\left[\operatorname{arctg}\frac{\lambda_p-\lambda_q+\delta}{\varepsilon}-\operatorname{arctg}\frac{\lambda_p-\lambda_q-\delta}{\varepsilon}\right]=\sum_{\substack{q\\ \lambda_q=\lambda_p}}{}'\bar{g}_q f_q,\end{aligned}\right\}\quad(1.4.23)$$

so that $E(\lambda_p+0)-E(\lambda_p-0)$ is the projection on the subspace of $\mathfrak{L}^2$ spanned by the functions ψ_q with $\lambda_q=\lambda_p$. It will be observed that in the above argument the limits with respect to ε and δ cannot be interchanged.

It is plausible after the foregoing that

$$(g,Hf)=\int_{-\infty}^{\infty} l\,dE(l;g,f),\qquad(1.4.24)$$

provided f belongs to $\mathfrak{D}(H)$, a condition which is expressed by

$$\int_{-\infty}^{\infty} l^2\,dE(l;f,f)<\infty.\qquad(1.4.25)$$

Equation (1.4.24) is discussed in great detail by ACHIESER and GLASMANN(2), section 66. It is another illustration of the fact that, roughly speaking, $E(l+\Delta l)-E(l)$ selects that part of f for which the expectation value of the energy is between l and $l+\Delta l$.

General functions of H are defined by (ref.(2) section 74)

$$(g,F(H)f)=\int_{-\infty}^{\infty}F(l)\,dE(l;g,f),\qquad(1.4.26)$$

provided f is such that

$$\int_{-\infty}^{\infty}|F(l)|^2\,dE(l;f,f)<\infty.\qquad(1.4.27)$$

The expansion of the resolvent, eq. (1.4.7), is a special example of eq. (1.4.26). Also,

$$(g,e^{iHt}f)=\int_{-\infty}^{\infty}e^{ilt}\,dE(l;g,f)=\lim_{\substack{L\to-\infty\\ T\to\infty}}\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{L}^{T}e^{ilt}(g,[R(l+i\varepsilon)-R(l-i\varepsilon)]f)\,dl,\qquad(1.4.28)$$

where the transition from the second to the third member is made with the Helly-Bray theorem, as in the proof of eq. (1.4.11).

If the resolvent can be represented with the help of a Green function with the symmetry property (1.3.17), it is useful to introduce

$$\operatorname{Im}G(\boldsymbol{x},\boldsymbol{y};\lambda)=\frac{1}{2i}[G(\boldsymbol{x},\boldsymbol{y};\lambda)-G(\boldsymbol{x},\boldsymbol{y};\bar{\lambda})].\qquad(1.4.29)$$

Then for any bounded and continuous function F the expansion formula may be written in the form

$$(g, F(H)f) = \lim_{\substack{L\to-\infty \\ T\to\infty}} \lim_{\varepsilon\to 0} \frac{1}{\pi}\int_L^T F(l)\,dl \int \bar{g}(\boldsymbol{x})\,d\boldsymbol{x} \int [\operatorname{Im} G(\boldsymbol{x}, \boldsymbol{y}; l+i\varepsilon)] f(\boldsymbol{y})\,d\boldsymbol{y}. \quad (1.4.30)$$

1.4.3. An upper bound for the resolvent

If the resolvent is regular in a region which includes an interval $L_1 \le l \le L_2$ on the real axis, we have

$$\lim_{\varepsilon\to 0} \int_{L_1}^{L_2} (g, [R(l+i\varepsilon) - R(l-i\varepsilon)] f)\,dl = 0. \quad (1.4.31)$$

In other words, the integral in the expansion formula receives contributions only from the part of the real axis where the resolvent is singular, that is from the spectrum of H. In particular, if the spectrum is bounded below, i.e. if there exists a finite L such that the resolvent is regular in the half-plane $\operatorname{Re}\lambda \le L$, there is no contribution from l-values with $l \le L$. Hence, in this case the limit with respect to L may be omitted from equations such as eq. (1.4.11), provided L is chosen sufficiently small.

If the lower bound of the spectrum is denoted by Λ, it follows from eq. (1.4.4) that $E(l; f, f) = \text{const.}$ in the interval $-\infty < l < \Lambda$. Together with eq. (1.4.5) this yields that, in fact, $E(l; f, f) = 0$ if $-\infty < l \le \Lambda$. Hence, the integral representation of the resolvent may be written in the form

$$(f, R(\lambda)f) = \int_\Lambda^\infty \frac{dE(l; f, f)}{l-\lambda}. \quad (1.4.32)$$

Equation (1.4.1) now gives

$$(R(\lambda)f, R(\lambda)f) = \int_\Lambda^\infty \frac{dE(m; f, f)}{(m-\lambda)(m-\bar{\lambda})}. \quad (1.4.33)$$

If in this expression we write $\lambda = \Lambda + le^{i\varphi}$, then for $\frac{\pi}{2} \le \varphi \le \frac{3}{2}\pi$ we get

$$\left.\begin{aligned} \|R(\Lambda + le^{i\varphi})f\|^2 &= \int_\Lambda^\infty [(m-\Lambda)^2 + l^2 - 2(m-\Lambda)\cos\varphi]^{-1} dE(m; f, f) \\ &\le \frac{1}{l^2}\int_\Lambda^\infty dE(m; f, f) = \frac{1}{l^2}\|f\|^2 \le \frac{1}{l^2 \sin^4 \frac{1}{2}\varphi}\|f\|^2. \end{aligned}\right\} \quad (1.4.34)$$

It is already known from eq. (1.3.29) that

$$\|R(\Lambda + le^{i\varphi})\| \le \frac{1}{|l\sin\varphi|}. \quad (1.4.35)$$

Combining these two inequalities shows that, whenever $0 < \varphi < 2\pi$,

$$\|R(\lambda)\| \le \frac{1}{[\operatorname{Im}\sqrt{(\lambda-\Lambda)}]^2}. \quad (1.4.36)$$

This is a generalization of eq. (1.3.31) for the norm of $R_0(\lambda)$.

1.5. A separable three-body problem

1.5.1. The resolvent

We are now able to tackle the simple three-body problem in which there is an interaction V_{12} between particles 1 and 2, but no interaction between particle 3 and particles 1 and 2. By eq. (1.2.4), the differential operator for the relative motion is in this case

$$\left.\begin{aligned} H'(\boldsymbol{x}_1, \boldsymbol{x}_2) &= H'_{12}(\boldsymbol{x}_1) + H'_0(\boldsymbol{x}_2), \\ H'_{12}(\boldsymbol{x}_1) = -\Delta(\boldsymbol{x}_1) + V_{12}(\boldsymbol{x}_1), &\quad H'_0(\boldsymbol{x}_2) = -\Delta(\boldsymbol{x}_2), \end{aligned}\right\} \tag{1.5.1}$$

the Hamiltonian being the self-adjoint extension of $H'(\boldsymbol{x}_1, \boldsymbol{x}_2)$.

The important point about eq. (1.5.1) is that H' separates into two differential operators which act in different Hilbert spaces. It is therefore useful to consider the spaces $\mathfrak{L}^2(\boldsymbol{x}_1)$, $\mathfrak{L}^2(\boldsymbol{x}_2)$, and $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$. The extension of $H'_{12}(\boldsymbol{x}_1)$ in $\mathfrak{L}^2(\boldsymbol{x}_1)$ is denoted by $H_{12}(\boldsymbol{x}_1)$, the corresponding resolvent by $R_{12}(\boldsymbol{x}_1; \lambda)$. Likewise, the extension of $H'_0(\boldsymbol{x}_2)$ in $\mathfrak{L}^2(\boldsymbol{x}_2)$ is denoted by $H_0(\boldsymbol{x}_2)$, the resolvent by $R_0(\boldsymbol{x}_2; \lambda)$.

In an obvious way the Hamiltonians $H_{12}(\boldsymbol{x}_1)$ and $H_0(\boldsymbol{x}_2)$ can be considered as operators not only in $\mathfrak{L}^2(\boldsymbol{x}_1)$ and $\mathfrak{L}^2(\boldsymbol{x}_2)$, respectively, but also in $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$. In this sense, we have

$$H(\boldsymbol{x}_1, \boldsymbol{x}_2) = H_{12}(\boldsymbol{x}_1) + H_0(\boldsymbol{x}_2). \tag{1.5.2}$$

Likewise, the resolvents can be considered as operators in $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$, with the same bounds as in $\mathfrak{L}^2(\boldsymbol{x}_1)$ and $\mathfrak{L}^2(\boldsymbol{x}_2)$, respectively. In the space $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ we write

$$(g(\boldsymbol{x}_1, \boldsymbol{x}_2), R_{12}(\boldsymbol{x}_1; \lambda) f(\boldsymbol{x}_1, \boldsymbol{x}_2)) = \int \bar{g}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, d\boldsymbol{x}_1\, d\boldsymbol{x}_2 \int G_{12}(\boldsymbol{x}_1, \boldsymbol{y}_1; \lambda) f(\boldsymbol{y}_1, \boldsymbol{x}_2)\, d\boldsymbol{y}_1, \tag{1.5.3}$$

and similarly for $R_0(\boldsymbol{x}_2; \lambda)$.

Now, if $\varphi_{12}(\boldsymbol{x}_1)$ is an eigenfunction of $H_{12}(\boldsymbol{x}_1)$, and $\varphi_0(\boldsymbol{x}_2)$ an eigenfunction of $H_0(\boldsymbol{x}_2)$, the product $\varphi_{12}(\boldsymbol{x}_1)\, \varphi_0(\boldsymbol{x}_2)$ is an eigenfunction of $H(\boldsymbol{x}_1, \boldsymbol{x}_2)$. And if the spectra of $H_{12}(\boldsymbol{x}_1)$ and $H_0(\boldsymbol{x}_2)$ were discrete, all the eigenfunctions of $H(\boldsymbol{x}_1, \boldsymbol{x}_2)$ could be obtained in this way. In view of the relation between resolvents, Green functions, and complete orthonormal sets of eigenfunctions brought out by the spectral theory, this suggests that, if the resolvents $R_{12}(\boldsymbol{x}_1; \lambda)$ and $R_0(\boldsymbol{x}_2; \lambda)$ are known, the resolvent $R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ can be constructed therefrom. It is the purpose of the present section to prove that this is indeed the case. As a matter of fact, it is shown that

$$\left.\begin{aligned} &(g(\boldsymbol{x}_1, \boldsymbol{x}_2), R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda) f(\boldsymbol{x}_1, \boldsymbol{x}_2)) \\ &= \frac{1}{2\pi i} \int_C (g(\boldsymbol{x}_1, \boldsymbol{x}_2), R_0(\boldsymbol{x}_2; \lambda - \sigma) R_{12}(\boldsymbol{x}_1; \sigma) f(\boldsymbol{x}_1, \boldsymbol{x}_2))\, d\sigma, \end{aligned}\right\} \tag{1.5.4}$$

where C is a suitable contour in the σ-plane such that the singularities of $R_{12}(\boldsymbol{x}_1; \sigma)$ are on the right of C, and those of $R_0(\boldsymbol{x}_2; \lambda - \sigma)$ on the left of C.

To be explicit, let us choose

$$\lambda = l + i\nu \ (\nu > 0), \quad \sigma = s + \frac{1}{2} i\nu \ (-\infty < s < \infty). \tag{1.5.5}$$

Then, denoting the right-hand side of eq. (1.5.4) by I, we have

$$\left.\begin{aligned} I \leq \frac{1}{2\pi} \Big[\int_{-\infty}^{\infty} || R_0\Big(\boldsymbol{x}_2;\, l - s - \frac{1}{2} i\nu\Big)\, g(\boldsymbol{x}_1, \boldsymbol{x}_2) ||^2 \, ds \Big]^{\frac{1}{2}} \\ \times \Big[\int_{-\infty}^{\infty} || R_{12}\Big(\boldsymbol{x}_1;\, s + \frac{1}{2} i\nu\Big) f(\boldsymbol{x}_1, \boldsymbol{x}_2) ||^2 \, ds \Big]^{\frac{1}{2}}. \end{aligned}\right\} \tag{1.5.6}$$

Our first step is now to establish that $I/(||f||\ ||g||)$ is bounded uniformly in λ in the half-plane $\operatorname{Im}\lambda \geq \varepsilon > 0$. To do this, we note that since $f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ belongs to $\mathfrak{L}^2(\boldsymbol{x}_1)$ for almost every $\boldsymbol{x}_2$, by Fubini's theorem, the spectral theory of section 1.4 can be applied to $R_{12}(\boldsymbol{x}_1; \sigma)$ by keeping $\boldsymbol{x}_2$ fixed at first, and integrating over $\boldsymbol{x}_2$ as a last step. In this way, eqs. (1.4.1) and (1.4.7) give

$$\left.\begin{aligned} I_{12} \equiv \int_{-\infty}^{\infty} || R_{12}\Big(\boldsymbol{x}_1;\, s + \frac{1}{2} i\nu\Big) f(\boldsymbol{x}_1, \boldsymbol{x}_2) ||^2 \, ds \\ = \int_{-\infty}^{\infty} ds \int d\boldsymbol{x}_2 \int \frac{d_m E(\boldsymbol{x}_1;\, m;\, f(\boldsymbol{x}_1, \boldsymbol{x}_2), f(\boldsymbol{x}_1, \boldsymbol{x}_2))}{(m-s)^2 + \frac{1}{4}\nu^2}, \end{aligned}\right\} \tag{1.5.7}$$

where the argument $\boldsymbol{x}_1$ of E denotes that the spectral theorem has been applied in the space $\mathfrak{L}^2(\boldsymbol{x}_1)$. The integration with respect to s may now be performed first, and a bound for I_{12} is obtained of the form

$$I_{12} \leq \frac{2\pi}{\nu} \int d\boldsymbol{x}_2 \, E(\boldsymbol{x}_1; \infty; f(\boldsymbol{x}_1, \boldsymbol{x}_2), f(\boldsymbol{x}_1, \boldsymbol{x}_2)) = \frac{2\pi}{\nu} \int d\boldsymbol{x}_2 \int |f(\boldsymbol{x}_1, \boldsymbol{x}_2)|^2 \, d\boldsymbol{x}_1 = \frac{2\pi}{\nu} ||f||^2. \tag{1.5.8}$$

As a similar argument applies to $R_0(\boldsymbol{x}_2; \lambda - \sigma)$, it follows that $I/(||f||\ ||g||)$ is bounded if $\nu \geq \varepsilon > 0$, as we wished to show. The useful consequence of this result is that it is sufficient to check the desired relation (1.5.4) for functions f and g in $\mathfrak{D}(H(\boldsymbol{x}_1, \boldsymbol{x}_2))$ only. The final result for all f and g in $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ follows with a limiting procedure such as used to get eq. (1.4.16).

Let us choose, then, functions f and g in $\mathfrak{D}(H(\boldsymbol{x}_1, \boldsymbol{x}_2))$. These belong also to $\mathfrak{D}(H_{12}(\boldsymbol{x}_1))$ and $\mathfrak{D}(H_0(\boldsymbol{x}_2))$. Let us further apply the operator

$$1 = R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)\,[H_{12}(\boldsymbol{x}_1) + H_0(\boldsymbol{x}_2) - \lambda] \tag{1.5.9}$$

to

$$R_0(\boldsymbol{x}_2; \lambda - \sigma)\, R_{12}(\boldsymbol{x}_1; \sigma). \tag{1.5.10}$$

This yields

$$R_0(\boldsymbol{x}_2; \lambda - \sigma) R_{12}(\boldsymbol{x}_1; \sigma) = R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)\,[R_0(\boldsymbol{x}_2; \lambda - \sigma) + R_{12}(\boldsymbol{x}_1; \sigma)]. \tag{1.5.11}$$

By eq. (1.4.12) we have

$$\left.\begin{aligned} &(g(\boldsymbol{x}_1,\boldsymbol{x}_2), R_{12}(\boldsymbol{x}_1;\sigma) f(\boldsymbol{x}_1,\boldsymbol{x}_2)) \\ &= -\frac{1}{\sigma}(g(\boldsymbol{x}_1,\boldsymbol{x}_2), f(\boldsymbol{x}_1,\boldsymbol{x}_2)) + \frac{1}{\sigma}(g(\boldsymbol{x}_1,\boldsymbol{x}_2), R_{12}(\boldsymbol{x}_1;\sigma) H_{12}(\boldsymbol{x}_1) f(\boldsymbol{x}_1,\boldsymbol{x}_2)). \end{aligned}\right\} \quad (1.5.12)$$

Since $H_{12}(\boldsymbol{x}_1) f(\boldsymbol{x}_1, \boldsymbol{x}_2)$ belongs to $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ by assumption, we may apply the spectral formula for the resolvent to the second term on the right-hand side of eq. (1.5.12). With this it follows that

$$\left.\begin{aligned} &\int_{-\infty}^{\infty} \frac{ds}{s+\frac{1}{2}i\nu} (g(\boldsymbol{x}_1,\boldsymbol{x}_2), R_{12}\left(\boldsymbol{x}_1; s+\frac{1}{2}i\nu\right) H_{12}(\boldsymbol{x}_1) f(\boldsymbol{x}_1,\boldsymbol{x}_2)) \\ &= \int_{-\infty}^{\infty} \frac{ds}{s+\frac{1}{2}i\nu} \int d\boldsymbol{x}_2 \int \frac{1}{m-s-\frac{1}{2}i\nu} d_m E(\boldsymbol{x}_1; m; g(\boldsymbol{x}_1,\boldsymbol{x}_2), H_{12}(\boldsymbol{x}_1) f(\boldsymbol{x}_1,\boldsymbol{x}_2)) = 0. \end{aligned}\right\} \quad (1.5.13)$$

Hence it is only the first term on the right-hand side of eq. (1.5.12) which gives a non-vanishing contribution to the convolution integral I we want to evaluate. A similar argument applies to the contributions from $R_0(\boldsymbol{x}_2; \lambda-\sigma)$. With eq. (1.5.11), I therefore takes the form

$$\left.\begin{aligned} I = -\frac{1}{2\pi i}\int_{-\infty}^{\infty} &\left(\frac{1}{l-s+\frac{1}{2}i\nu} + \frac{1}{s+\frac{1}{2}i\nu}\right)(g(\boldsymbol{x}_1,\boldsymbol{x}_2), R(\boldsymbol{x}_1,\boldsymbol{x}_2; l+i\nu) f(\boldsymbol{x}_1,\boldsymbol{x}_2))\, ds \\ &= (g(\boldsymbol{x}_1,\boldsymbol{x}_2), R(\boldsymbol{x}_1,\boldsymbol{x}_2;\lambda) f(\boldsymbol{x}_1,\boldsymbol{x}_2)), \end{aligned}\right\} \quad (1.5.14)$$

which establishes eq. (1.5.4) for $\operatorname{Im}\lambda > 0$. It can likewise be established for $\operatorname{Im}\lambda < 0$.

In the case at hand, in which the spectra involved are bounded below, the contour in eq. (1.5.4) may be deformed in various ways. If, for instance, $\lambda = l + i\nu$ $(\nu > 0)$ and

$$\sigma = \lambda + Te^{i\varphi} \quad (0 \le \varphi < \pi), \quad (1.5.15)$$

it follows from what we know about the bounds for R_{12} and R_0 that the integrand in eq. (1.5.4) does not exceed

$$\|g\|\,\|f\| \frac{1}{[\operatorname{Im}\sqrt{(\lambda-\sigma)}]^2} \frac{1}{\operatorname{Im}\sigma} = \|g\|\,\|f\| \frac{1}{T\cos^2\frac{1}{2}\varphi} \frac{1}{\nu + T\sin\varphi}. \quad (1.5.16)$$

Hence, if we integrate from $\varphi = 0$ to $\varphi = \varphi_0 < \pi$ along the circumference of the circle with centre λ and radius T, the integral so obtained tends to 0 as T tends to ∞. Likewise, denoting the lower bound of the spectrum of $H_{12}(\boldsymbol{x}_1)$ by Λ_{12}, we see that if

$$\sigma = \Lambda_{12} - T + i\tau \quad (\nu < \nu + \varepsilon \le \tau \le T), \quad (1.5.17)$$

the integrand does not exceed

$$\|g\|\,\|f\|\,\frac{1}{|\operatorname{Im}(\lambda-\sigma)|}\,\frac{1}{[\operatorname{Im}\sqrt{(\sigma-\Lambda_{12})}]^2} \le \|g\|\,\|f\|\,\frac{1}{\tau-\nu}\,\frac{1}{T}. \tag{1.5.18}$$

Hence the integral along the straight line from $\Lambda_{12}-T+i\nu+i\varepsilon$ to $\Lambda_{12}-T+iT$ also tends to 0 as T tends to ∞. With arguments of this sort it is easily shown that as our contour we may in fact take any path which does not cross the cuts in the σ-plane caused by the singularities of $R_{12}(\sigma)$ and $R_0(\lambda-\sigma)$.

If λ is in the half-plane $\operatorname{Re}\lambda \le \Lambda_{12}-\varepsilon < \Lambda_{12}$, it is convenient to choose the contour parallel to the imaginary axis. It is easily seen that the integral so obtained converges uniformly with respect to λ. Hence, since the integrand is a regular function of λ, so is the integral. If λ is not real, the integral yields the resolvent, by our previous analysis. Its being regular therefore implies that the resolvent exists, and is regular, throughout the half-plane $\operatorname{Re}\lambda < \Lambda_{12}$. Combining this with previous results shows that the resolvent exists everywhere in the λ-plane cut from Λ_{12} to ∞, and that it is a regular function which can be evaluated by means of a convolution integral according to eq. (1.5.4).

For future reference it is useful to write the right-hand side of eq. (1.5.4) out in the form

$$\frac{1}{2\pi i}\int_C d\sigma \int \bar{g}\,(\boldsymbol{x}_1,\boldsymbol{x}_2)\,R_0(\boldsymbol{x}_2;\,\lambda-\sigma)\,R_{12}(\boldsymbol{x}_1;\,\sigma)\,f(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2, \tag{1.5.19}$$

and to compare this expression with

$$\left.\begin{aligned}&\frac{1}{2\pi}\int_C d\sigma \int |\,\bar{g}\,(\boldsymbol{x}_1,\boldsymbol{x}_2)\,R_0(\boldsymbol{x}_2;\,\lambda-\sigma)\,R_{12}(\boldsymbol{x}_1;\,\sigma)\,f(\boldsymbol{x}_1,\boldsymbol{x}_2)\,|\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\\ &\qquad\le \frac{1}{2\pi}\int_C d\sigma\,\|g\|\,\|f\|\,\frac{1}{[\operatorname{Im}\sqrt{(\lambda-\sigma)}]^2}\,\frac{1}{[\operatorname{Im}\sqrt{(\sigma-\Lambda_{12})}]^2}.\end{aligned}\right\} \tag{1.5.20}$$

If C is chosen in a suitable way, this integral is certainly convergent. Now in the original convolution expression, $\int_C d\sigma$ was meant to be a Riemann integral. However, if we are willing to interpret it as a Lebesgue integral, it becomes possible to invoke Fubini's theorem to invert the order of integration in the expression (1.5.19). Then we may write

$$R(\boldsymbol{x}_1,\boldsymbol{x}_2;\,\lambda)\,f(\boldsymbol{x}_1,\boldsymbol{x}_2) = \frac{1}{2\pi i}\int_C R_0(\boldsymbol{x}_2;\,\lambda-\sigma)\,R_{12}(\boldsymbol{x}_1;\,\sigma)\,f(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\sigma. \tag{1.5.21}$$

In this sense there is a convolution integral not only for the inner product $(g,\,R(\lambda)f)$, but also for the function $R(\lambda)f$.

1.5.2. The Green function

The object of the present section is to show that the resolvent $R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ is an integral operator the kernel of which is a Green function of the form

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) = \frac{1}{2\pi i}\int_C G_0(\boldsymbol{x}_2, \boldsymbol{y}_2; \lambda - \sigma)\, G_{12}(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma)\, d\sigma. \qquad (1.5.22)$$

The distinction between this and the results (1.5.4) and (1.5.21) concerning the resolvent is that in the latter expressions certain integrations with respect to the space coodinates are performed first, the integration with respect to σ next, whereas in eq. (1.5.22) it is implied that the integration with respect to σ must be performed first.

To prove eq. (1.5.22), we split G_{12} according to eq. (1.3.27),

$$G_{12}(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma) = G_0(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma) - \int G_0(\boldsymbol{x}_1, \boldsymbol{z}_1; \sigma)\, V_{12}(\boldsymbol{z}_1)\, G_{12}(\boldsymbol{z}_1, \boldsymbol{y}_1; \sigma)\, d\boldsymbol{z}_1. \qquad (1.5.23)$$

This corresponds to a splitting of R of the form

$$R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda) = R_0(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda) - [R_0(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda) - R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)]. \qquad (1.5.24)$$

Here it is known from eq. (1.2.17) that R_0 is an integral operator the kernel of which involves a Hankel function of order 2. In other words, there is a Green function

$$G_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) = \frac{i\lambda}{16\pi^2} \frac{1}{|\boldsymbol{x}_1 - \boldsymbol{y}_1|^2 + |\boldsymbol{x}_2 - \boldsymbol{y}_2|^2} H_2^{(1)}(\sqrt{\lambda}[|\boldsymbol{x}_1 - \boldsymbol{y}_1|^2 + |\boldsymbol{x}_2 - \boldsymbol{y}_2|^2]^{\frac{1}{2}}). \qquad (1.5.25)$$

It is now shown first that this function is obtained when the first term in the right-hand member of eq. (1.5.23) is folded according to eq. (1.5.22). More generally, it is shown that, when λ is not on the positive real axis and $r_1 \neq 0$, $r_2 \neq 0$,

$$\left.\begin{aligned} &-\frac{1}{32\pi i}(2\pi r_1)^{-p}(2\pi r_2)^{-q}\int_C \sigma^{\frac{1}{2}p}(\lambda - \sigma)^{\frac{1}{2}q} H_p^{(1)}(r_1\sqrt{\sigma})\, H_q^{(1)}(r_2\sqrt{(\lambda - \sigma)})\, d\sigma \\ &= \frac{i}{4}\left[\frac{\sqrt{\lambda}}{2\pi(r_1^2 + r_2^2)^{\frac{1}{2}}}\right]^{p+q+1} H_{p+q+1}^{(1)}((r_1^2 + r_2^2)^{\frac{1}{2}}\sqrt{\lambda}) \qquad \left(p, q = \frac{1}{2}, 1, \frac{3}{2}, \ldots\right). \end{aligned}\right\} \qquad (1.5.26)$$

Then it follows with eq. (1.2.17) that

$$\frac{1}{2\pi i}\int_C G_0^{(n_2)}(\boldsymbol{x}_2, \boldsymbol{y}_2; \lambda - \sigma)\, G_0^{(n_1)}(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma)\, d\sigma = G_0^{(n_1 + n_2 - 1)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda). \qquad (1.5.27)$$

In this relation the superscripts n refer to the numbers of particles involved. To describe the relative motion of n particles, $3n - 3$ coordinates are required. In particular, for three particles we need six coordinates. Three of these are used for the relative motion of particles 1 and 2, the remaining three for the motion of 3 with respect to the centre

of mass of 1 and 2. Since formally the latter motion is also of the two-particle type, it is consistent that the three-particle function $G_0^{(3)}$ is obtained from a convolution of two two-particle functions $G_0^{(2)}$. More generally, it is indeed the function for n_1+n_2-1 particles we must expect on the right-hand side of eq. (1.5.27), rather than for n_1+n_2 particles.

In evaluating the integral in eq. (1.5.26), it is convenient to deform the contour C into the positive real axis described twice in opposite directions, the origin being encircled clockwise. Then the integral takes the form

$$-\frac{1}{16\pi i}(2\pi r_1)^{-p}(2\pi r_2)^{-q}\int_0^\infty s^{\frac{1}{2}p}(\lambda-s)^{\frac{1}{2}q}J_p(r_1\sqrt{s})\,H_q^{(1)}(r_2\sqrt{(\lambda-s)})\,ds. \tag{1.5.28}$$

If we now make the substitution

$$H_q^{(1)}(r_2\sqrt{(\lambda-s)}) = \frac{2}{\pi i}e^{-\frac{1}{2}q\pi i}K_{-q}(r_2\sqrt{(s-\lambda)}), \tag{1.5.29}$$

where $\sqrt{(s-\lambda)}$ must be chosen such that its real part is positive, we get an integral of the type studied by SONINE and GEGENBAUER. According to WATSON (19), section 13.47, it is proportional to $K_{-p-q-1}(-i(r_1^2+r_2^2)^{\frac{1}{2}}\sqrt{\lambda})$. If this K-function is again expressed in terms of a Hankel function, the desired relation (1.5.26) immediately follows.

It remains to compare the second terms on the right-hand sides of eqs. (1.5.23) and (1.5.24). According to eq. (1.5.4) and the known representations of $R_0(\boldsymbol{x}_2;\lambda-\sigma)$ and $R_{12}(\boldsymbol{x}_1;\sigma)$, we have

$$\left.\begin{aligned}&(g(\boldsymbol{x}_1,\boldsymbol{x}_2),[R_0(\boldsymbol{x}_1,\boldsymbol{x}_2;\lambda)-R(\boldsymbol{x}_1,\boldsymbol{x}_2;\lambda)]f(\boldsymbol{x}_1,\boldsymbol{x}_2))\\ =&\frac{1}{2\pi i}\int_C d\sigma\int\bar{g}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int G_0(\boldsymbol{x}_2,\boldsymbol{y}_2;\lambda-\sigma)\,[G_0(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)-G_{12}(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)]\,f(\boldsymbol{y}_1,\boldsymbol{y}_2)\,d\boldsymbol{y}_1\,d\boldsymbol{y}_2\\ =&\frac{1}{2\pi i}\int_C d\sigma\int\bar{g}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int G_0(\boldsymbol{x}_2,\boldsymbol{y}_2;\lambda-\sigma)f(\boldsymbol{y}_1,\boldsymbol{y}_2)\,d\boldsymbol{y}_1\,d\boldsymbol{y}_2\int G_0(\boldsymbol{x}_1,\boldsymbol{z}_1;\sigma)V_{12}(\boldsymbol{z}_1)\,G_{12}(\boldsymbol{z}_1,\boldsymbol{y}_1;\sigma)\,d\boldsymbol{z}_1,\end{aligned}\right\} \tag{1.5.30}$$

the third member following from eq. (1.3.27). To establish $R(\boldsymbol{x}_1,\boldsymbol{x}_2;\lambda)$ as an integral operator, it must be shown that in eq. (1.5.30)

$$\int d\sigma\int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int d\boldsymbol{y}_1\,d\boldsymbol{y}_2 = \int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int d\boldsymbol{y}_1\,d\boldsymbol{y}_2\int d\sigma. \tag{1.5.31}$$

Writing

$$F(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma) = -\int G_0(\boldsymbol{x}_1,\boldsymbol{z}_1;\sigma)\,V_{12}(\boldsymbol{z}_1)\,G_{12}(\boldsymbol{z}_1,\boldsymbol{y}_1;\sigma)\,d\boldsymbol{z}_1, \tag{1.5.32}$$

we know from eqs. (1.3.4) and (1.4.36) that

$$\iint|F(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)|^2\,d\boldsymbol{x}_1\,d\boldsymbol{y}_1 \le |R_0(\sigma)V_{12}|^2\,\|R_{12}(\sigma)\|^2 \le \text{const.}\,\frac{1}{\operatorname{Im}\sqrt{\sigma}}\,\frac{1}{[\operatorname{Im}\sqrt{(\sigma-\Lambda_{12})}]^4}. \tag{1.5.33}$$

Substituting the expression for $G_0(\boldsymbol{x}_2, \boldsymbol{y}_2; \lambda-\sigma)$ and introducing $\boldsymbol{x}_2 - \boldsymbol{y}_2 = \boldsymbol{y}_2'$ now yields

$$\left.\begin{aligned}
&\int_C d\sigma \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2 \int d\boldsymbol{y}_1\, d\boldsymbol{y}_2 \,|\, \bar{g}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, G_0(\boldsymbol{x}_2, \boldsymbol{y}_2; \lambda-\sigma)\, F(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma)\, f(\boldsymbol{y}_1, \boldsymbol{y}_2)\,| \\
&= \frac{1}{4\pi} \int_C d\sigma \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2 \int d\boldsymbol{y}_1\, d\boldsymbol{y}_2' \left| \bar{g}(\boldsymbol{x}_1, \boldsymbol{x}_2)\, \frac{e^{i\sqrt{\lambda-\sigma}\,|\boldsymbol{y}_2'|}}{|\boldsymbol{y}_2'|}\, F(\boldsymbol{x}_1, \boldsymbol{y}_1; \sigma)\, f(\boldsymbol{y}_1, \boldsymbol{x}_2 - \boldsymbol{y}_2') \right| \\
&\leq \frac{1}{4\pi} \int_C d\sigma \int d\boldsymbol{y}_2' \left| \frac{e^{i\sqrt{\lambda-\sigma}\,|\boldsymbol{y}_2'|}}{|\boldsymbol{y}_2'|} \right| \|g\|\, |F(\sigma)|\, \|f\| \\
&\leq \text{const.} \int_C d\sigma\, \frac{1}{[\operatorname{Im}\sqrt{(\lambda-\sigma)}]^2}\, \frac{1}{(\operatorname{Im}\sqrt{\sigma})^{\frac{1}{2}}}\, \frac{1}{[\operatorname{Im}\sqrt{(\sigma-\Lambda_{12})}]^2}\, \|g\|\, \|f\|,
\end{aligned}\right\} \tag{1.5.34}$$

which is certainly finite if C is a suitable contour. Hence, if we interpret $\int_C d\sigma$ as a Lebesgue integral once more, as was done at the end of the previous section, we may invert the order of integration according to eq. (1.5.31) by Fubini's theorem. Then it follows that the resolvent is indeed an integral operator, the kernel of which can be evaluated by means of a convolution integral according to eq. (1.5.22).

1.5.3. Uniqueness of the Green function

It follows from the general theory of Hilbert space that the resolvent is unique. However, this does not imply that it can be represented by a Green function in only one single way. Let us therefore first consider the problem of the uniqueness of the function G_0. According to eq. (1.2.17), there is a function G_0 which depends only on $\boldsymbol{x}-\boldsymbol{y}$, and which is an integrable function of this difference. Let us now imagine that there are two functions G_0 with these properties, and let us denote their difference by Q_0. Then Q_0 satisfies

$$\int |\, Q_0(\boldsymbol{x}-\boldsymbol{y})\,|\, d(\boldsymbol{x}-\boldsymbol{y}) < \infty. \tag{1.5.35}$$

Also, by the uniqueness of the resolvent,

$$\int \bar{g}(\boldsymbol{x})\, d\boldsymbol{x} \int Q_0(\boldsymbol{x}-\boldsymbol{y})\, f(\boldsymbol{y})\, d\boldsymbol{y} = 0 \tag{1.5.36}$$

for every f and g in $\mathfrak{L}^2$. Writing $\boldsymbol{x}-\boldsymbol{y} = \boldsymbol{z}$ and going over to Fourier transforms $\hat{f}$ and $\hat{g}$ yields

$$\iint \bar{\hat{g}}(\boldsymbol{k})\, Q_0(\boldsymbol{z})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\, \hat{f}(\boldsymbol{k})\, d\boldsymbol{z}\, d\boldsymbol{k} = 0. \tag{1.5.37}$$

In view of the arbitrariness of $\hat{f}$ and $\hat{g}$, it now follows that

$$\int Q_0(\boldsymbol{z})\, e^{-i\boldsymbol{k}\cdot\boldsymbol{z}}\, d\boldsymbol{z} = 0 \tag{1.5.38}$$

for almost every $\boldsymbol{k}$. But this result can easily be extended. For, owing to eq. (1.5.35), the integral in eq. (1.5.38) is absolutely convergent. It is easily shown that as a consequence it is a continuous function of $\boldsymbol{k}$. Hence it vanishes not only for almost every $\boldsymbol{k}$, but in fact for every $\boldsymbol{k}$. From this it now follows that $Q_0(\boldsymbol{z})$ vanishes for almost every $\boldsymbol{z}$ (TITCHMARSH(20) section 6.7), so that the Green function $G_0(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ is in fact unique among all integrable functions of $\boldsymbol{x}-\boldsymbol{y}$. It will be observed that this conclusion does not depend on the number of particles involved.

We now turn to the function G. After the foregoing, it suffices to consider the difference $G-G_0$. It was already used at the end of the preceding section that this depends only on $\boldsymbol{x}_1, \boldsymbol{y}_1$, and $\boldsymbol{x}_2-\boldsymbol{y}_2$, and that it satisfies

$$\int \left[\int | G(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) - G_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) | \, d(\boldsymbol{x}_2-\boldsymbol{y}_2) \right]^2 d\boldsymbol{x}_1 \, d\boldsymbol{y}_1 < \infty. \quad (1.5.39)$$

Let us now again assume that there are two functions $G-G_0$ with these properties, and let us denote their difference by $Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{x}_2-\boldsymbol{y}_2)$. This gives

$$\int \bar{g}(\boldsymbol{x}_1, \boldsymbol{x}_2) \, d\boldsymbol{x}_1 \, d\boldsymbol{x}_2 \int Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{x}_2-\boldsymbol{y}_2) f(\boldsymbol{y}_1, \boldsymbol{y}_2) \, d\boldsymbol{y}_1 \, d\boldsymbol{y}_2 = 0. \quad (1.5.40)$$

Choosing in particular

$$f(\boldsymbol{x}_1, \boldsymbol{x}_2) = a(\boldsymbol{x}_1) f(\boldsymbol{x}_2), \quad g(\boldsymbol{x}_1, \boldsymbol{x}_2) = b(\boldsymbol{x}_1) g(\boldsymbol{x}_2), \quad (1.5.41)$$

we easily get an equation analogous to eq. (1.5.37), viz.

$$\int\int \bar{\hat{g}}(\boldsymbol{k}) \left[\int\int \bar{b}(\boldsymbol{x}_1) Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}) a(\boldsymbol{y}_1) \, d\boldsymbol{x}_1 \, d\boldsymbol{y}_1 \right] e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} \hat{f}(\boldsymbol{k}) \, d\boldsymbol{z} \, d\boldsymbol{k} = 0. \quad (1.5.42)$$

As above, this implies that

$$\int e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} d\boldsymbol{z} \int\int \bar{b}(\boldsymbol{x}_1) Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}) a(\boldsymbol{y}_1) \, d\boldsymbol{x}_1 \, d\boldsymbol{y}_1 = 0 \quad (1.5.43)$$

for almost every $\boldsymbol{k}$, and even for every $\boldsymbol{k}$. From the arbitrariness of a and b it now follows that

$$\int Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}) e^{-i\boldsymbol{k}\cdot\boldsymbol{z}} \, d\boldsymbol{z} = 0 \quad (1.5.44)$$

identically in $\boldsymbol{k}$ for almost every $\boldsymbol{x}_1, \boldsymbol{y}_1$. Also, the integral in eq. (1.5.44) converges absolutely for almost every $\boldsymbol{x}_1, \boldsymbol{y}_1$, by Fubini's theorem and the integrability properties of Q. Hence, for almost every $\boldsymbol{x}_1, \boldsymbol{y}_1$ we have an integrable function of $\boldsymbol{z}$, the Fourier transform of which vanishes identically. Since such a function is known to vanish for almost every $\boldsymbol{z}$, it follows that $Q(\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}) = 0$ for almost every $\boldsymbol{x}_1, \boldsymbol{y}_1, \boldsymbol{z}$. Hence, among all the functions with integrability properties as indicated by the relation (1.5.39), the function $G-G_0$ is unique, with the trivial exception of sets of measure zero. In particular, $G-G_0$ does not depend on the choice of the contour C, as long as $\int_C d\sigma$ converges properly. This was plausible from the outset. But it was not obvious,

say from an argument about contour integration, as we have not discussed the analytic properties of functions which depend not only on λ, but also on variables $\boldsymbol{x}$ and $\boldsymbol{y}$.

It follows from eqs. (1.5.22) and (1.3.17) that

$$\overline{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) = -\frac{1}{2\pi i}\int_{\overline{C}} G_0(\boldsymbol{x}_2, \boldsymbol{y}_2; \bar{\lambda}-\bar{\sigma})\, G_{12}(\boldsymbol{x}_1, \boldsymbol{y}_1; \bar{\sigma})\, d\bar{\sigma}, \tag{1.5.45}$$

where $\overline{C}$ is the reflection of C in the real axis. Now the right-hand side of eq. (1.5.45) is equal to a function of the form $G(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \bar{\lambda})$. Hence, since the Green function is unique,

$$\overline{G}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) = G(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \bar{\lambda}). \tag{1.5.46}$$

Also, by eq. (1.3.18),

$$G(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \lambda) = G(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{x}_1, \boldsymbol{x}_2; \lambda). \tag{1.5.47}$$

1.6. The Green function for three particles

1.6.1. The resolvent equation

After the preparations made in the previous sections, it is comparatively easy to find the Green function for a system of three particles in which there are non-vanishing interactions V_{12}, V_{13}, and V_{23}. With the choice of coordinates discussed in section 1.2.1, these are functions

$$V_{12} = V_{12}(c_{12}^1 \boldsymbol{x}_1), \quad V_{13} = V_{13}(c_{13}^1 \boldsymbol{x}_1 + c_{13}^2 \boldsymbol{x}_2), \quad V_{23} = V_{23}(c_{23}^1 \boldsymbol{x}_1 + c_{23}^2 \boldsymbol{x}_2), \tag{1.6.1}$$

where it is essential that the constants c_{ij}^k do not vanish. To save writing, we define $V_{ji} \equiv V_{ij}$ $(i<j)$. The resolvent for the present problem is denoted by $R_{123}^{(3)}$, the upper index indicating that three particles are involved, and the lower index that all the three interactions are present. The resolvent for the three-particle system without interaction is henceforth denoted by $R_0^{(3)}$, the one with interaction V_{ij} only by $R_{ij}^{(3)}$. There is no need to consider a three-particle system with two interaction terms such as $V_{ij} + V_{ik}$.

From the considerations which led to the resolvent equation (1.2.14), it is easily seen that $R_{123}^{(3)}$ satisfies

$$R_{123}^{(3)}(\lambda) = R_{ij}^{(3)}(\lambda) - R_0^{(3)}(\lambda)\Big(\sum_{i<j} V_{ij}\Big) R_{123}^{(3)}(\lambda), \tag{1.6.2}$$

as well as the three equations of the form

$$R_{123}^{(3)}(\lambda) = R_{ij}^{(3)}(\lambda) - R_{ij}^{(3)}(\lambda)(V_{ik} + V_{jk}) R_{123}^{(3)}(\lambda) \quad (i<j,\ i\neq k,\ j\neq k). \tag{1.6.3}$$

Also,

$$R_{ij}^{(3)}(\lambda) = R_0^{(3)}(\lambda) - R_0^{(3)}(\lambda) V_{ij} R_{ij}^{(3)}(\lambda). \tag{1.6.4}$$

Adding the three equations (1.6.3), subtracting twice equation (1.6.2) and using the relations (1.6.4) yields

$$\left.\begin{aligned} R^{(3)}_{123}(\lambda)f(\boldsymbol{x}) &= \sum_{i<j} R^{(3)}_{ij}(\lambda)f(\boldsymbol{x}) - 2\,R^{(3)}_{0}(\lambda)f(\boldsymbol{x}) + \sum_{\substack{i<j\\ i\neq k,\, j\neq k}} [R^{(3)}_{0}(\lambda) - R^{(3)}_{ij}(\lambda)]\,(V_{ik}+V_{jk})\,R^{(3)}_{123}(\lambda)f(\boldsymbol{x}) \\ &= \sum_{i<j} R^{(3)}_{ij}(\lambda)f(\boldsymbol{x}) - 2\,R^{(3)}_{0}(\lambda)f(\boldsymbol{x}) + \sum_{\substack{i<j\\ i\neq k,\, j\neq k}} R^{(3)}_{0}(\lambda)\,V_{ij}\,R^{(3)}_{ij}(\lambda)\,(V_{ik}+V_{jk})\,R^{(3)}_{123}(\lambda)f(\boldsymbol{x}). \end{aligned}\right\} \quad (1.6.5)$$

The point is now that this can be written out as an integral equation for $R^{(3)}_{123}(\lambda)\,f(\boldsymbol{x})$ the kernel of which will be shown to belong to $\mathfrak{L}^2$. This means that from the Hilbert-space point of view eq. (1.6.5) is entirely analogous to the two-particle equation (1.3.2). In particular, it can be solved by the Fredholm technique. And owing to our knowledge of the bounds and the analyticity properties of the resolvents $R^{(3)}_0$ and $R^{(3)}_{ij}$, the symmetries of the associated Green functions, and related matters, all considerations of section 1.3 can be carried over directly to the present problem. We thus find a Green function of the form

$$\left.\begin{aligned} G^{(3)}_{123}(\boldsymbol{x},\boldsymbol{y};\,\lambda) &= \sum_{i<j} G^{(3)}_{ij}(\boldsymbol{x},\boldsymbol{y};\,\lambda) - 2\,G^{(3)}_{0}(\boldsymbol{x},\boldsymbol{y};\,\lambda) \\ &+ \frac{1}{\Delta^{(3)}_{123}(\lambda)} \int D^{(3)}_{123}(\boldsymbol{x},\boldsymbol{z};\,\lambda)\Big[\sum_{i<j} G^{(3)}_{ij}(\boldsymbol{z},\boldsymbol{y};\,\lambda) - 2\,G^{(3)}_{0}(\boldsymbol{z},\boldsymbol{y};\,\lambda)\Big]\,d\boldsymbol{z}, \end{aligned}\right\} \quad (1.6.6)$$

which is entirely analogous to the two-particle Green function discussed in section 1.3. It is obvious that in eq. (1.6.6) $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$ are meant to be six-dimensional coordinates. From the properties of the resolvents $R^{(3)}_{ij}$ it follows that in the three-particle case there is a cut in the λ-plane from $\min(\Delta_{12}, \Delta_{13}, \Delta_{23})$ to ∞, Δ_{ij} being the lower bound of the two-particle spectrum concerned. In the cut plane, the resolvent $R^{(3)}_{123}(\lambda)$ is regular except for possible poles.

1.6.2. An upper bound for the kernel

To return to the equation we have to solve, eq. (1.6.5) is of the form

$$\left.\begin{aligned} h^{(3)}_{123}(\boldsymbol{x};\,\lambda) &= \sum_{i<j} h^{(3)}_{ij}(\boldsymbol{x};\,\lambda) - 2\,h^{(3)}_{0}(\boldsymbol{x};\,\lambda) + \int K^{(3)}_{123}(\boldsymbol{x},\boldsymbol{y};\,\lambda)\,h^{(3)}_{123}(\boldsymbol{y};\,\lambda)\,d\boldsymbol{y}, \\ h^{(3)}_{123}(\boldsymbol{x};\,\lambda) &\equiv R^{(3)}_{123}(\lambda)f(\boldsymbol{x}), \\ h^{(3)}_{ij}(\boldsymbol{x};\,\lambda) &\equiv R^{(3)}_{ij}(\lambda)f(\boldsymbol{x}) = \int G^{(3)}_{ij}(\boldsymbol{x},\boldsymbol{z};\,\lambda)\,f(\boldsymbol{z})\,d\boldsymbol{z}, \\ h^{(3)}_{0}(\boldsymbol{x};\,\lambda) &\equiv R^{(3)}_{0}(\lambda)f(\boldsymbol{x}) = \int G^{(3)}_{0}(\boldsymbol{x},\boldsymbol{z};\,\lambda)\,f(\boldsymbol{z})\,d\boldsymbol{z}, \\ K^{(3)}_{123}(\boldsymbol{x},\boldsymbol{y};\,\lambda) &= \sum_{\substack{i<j\\ i\neq k,\, j\neq k}} [G^{(3)}_{0}(\boldsymbol{x},\boldsymbol{y};\,\lambda) - G^{(3)}_{ij}(\boldsymbol{x},\boldsymbol{y};\,\lambda)]\,[V_{ik}(\boldsymbol{y}) + V_{jk}(\boldsymbol{y})]. \end{aligned}\right\} \quad (1.6.7)$$

It is now shown that each of the three terms in the expression for $K^{(3)}_{123}$ belongs to $\mathfrak{L}^2$. Then the same applies to $K^{(3)}_{123}$ itself. In particular, let us consider the term with $i, j = 1, 2$. For this it follows from eqs. (1.5.22) and (1.5.23) that

$$\left.\begin{aligned} & G_0^{(3)}(\boldsymbol{x},\boldsymbol{y};\,\lambda) - G_{12}^{(3)}(\boldsymbol{x},\boldsymbol{y};\,\lambda) \\ &= \frac{1}{2\pi i}\int_C G_0^{(2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\,\lambda-\sigma)\,d\sigma \int G_0^{(2)}(\boldsymbol{x}_1,\boldsymbol{z}_1;\,\sigma)\,V_{12}(c_{12}^1\boldsymbol{z}_1)\,G_{12}^{(2)}(\boldsymbol{z}_1,\boldsymbol{y}_1;\,\sigma)\,d\boldsymbol{z}_1 . \end{aligned}\right\} \tag{1.6.8}$$

Repeated application of Schwarz's inequality yields

$$\left.\begin{aligned} I &\equiv \left[\int |G_0^{(3)}(\boldsymbol{x},\boldsymbol{y};\,\lambda) - G_{12}^{(3)}(\boldsymbol{x},\boldsymbol{y};\,\lambda)|^2\,d\boldsymbol{x}_1\,d\boldsymbol{y}_1\,d(\boldsymbol{x}_2-\boldsymbol{y}_2)\right]^{\frac{1}{2}} \\ &\leq \frac{1}{2\pi}\left\{\int_C d\sigma\left[\int |G_0^{(2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\,\lambda-\sigma)|^2\,d(\boldsymbol{x}_2-\boldsymbol{y}_2)\right]^{\frac{1}{2}}\right. \\ &\times \left.\left[\int\left|\int G_0^{(2)}(\boldsymbol{x}_1,\boldsymbol{z}_1;\,\sigma)\,V_{12}(c_{12}^1\boldsymbol{z}_1)\,G_{12}^{(2)}(\boldsymbol{z}_1,\boldsymbol{y}_1;\,\sigma)\,d\boldsymbol{z}_1\right|^2 d\boldsymbol{x}_1\,d\boldsymbol{y}_1\right]^{\frac{1}{2}}\right\} . \end{aligned}\right\} \tag{1.6.9}$$

With the relation

$$4\pi\int\left|\frac{e^{ir\sqrt{\lambda-\sigma}}}{4\pi r}\right|^2 r^2\,dr \leq \frac{1}{8\pi\,\mathrm{Im}\,\sqrt{(\lambda-\sigma)}} \tag{1.6.10}$$

to be used in connection with $G_0^{(2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\,\lambda-\sigma)$, and eqs. (1.5.32) and (1.5.33) for the second factor on the right-hand side of eq. (1.6.9), the inequality (1.6.9) can be reduced to

$$\left.\begin{aligned} I &\leq \text{const.}\int_C \frac{1}{[\mathrm{Im}\,\sqrt{(\lambda-\sigma)}]^{\frac{1}{2}}}\,\frac{1}{[\mathrm{Im}\,\sqrt{(\sigma-\Lambda_{12})}]^{\frac{5}{2}}}\,d\sigma \\ &\leq \text{const.}\left[\int_C \frac{d\sigma}{[\mathrm{Im}\,\sqrt{(\lambda-\sigma)}]^3}\right]^{\frac{1}{6}}\left[\int_C \frac{d\sigma}{[\mathrm{Im}\,\sqrt{(\sigma-\Lambda_{12})}]^3}\right]^{\frac{5}{6}}, \end{aligned}\right\} \tag{1.6.11}$$

the third member of eq. (1.6.11) following from Hölder's inequality (TITCHMARSH(15) section 12.42). In passing from eq. (1.5.33) to eq. (1.6.11), use was made of the relation

$$\mathrm{Im}\,\sqrt{(\sigma-\Lambda_{12})} \leq \mathrm{Im}\,\sqrt{\bar{\sigma}}, \tag{1.6.12}$$

which holds true by virtue of Λ_{12} being real and non-positive.

To find an upper bound for I, it is now convenient to write

$$\lambda-\Lambda_{12} = l e^{i\varphi} \tag{1.6.13}$$

and to choose as the contour C the straight line

$$\sigma = s e^{\frac{1}{2}i\varphi} - \frac{1}{2}l + \Lambda_{12} \quad (-\infty < s < \infty) . \tag{1.6.14}$$

This line is tangent to the parabolas $[\mathrm{Im}\,\sqrt{(\sigma-\Lambda_{12})}]^2 = \frac{1}{2}[\mathrm{Im}\,\sqrt{(\lambda-\Lambda_{12})}]^2$ and $[\mathrm{Im}\,\sqrt{(\lambda-\sigma)}]^2 = \frac{1}{2}[\mathrm{Im}\,\sqrt{(\lambda-\Lambda_{12})}]^2$. These parabolas in turn are tangent to one

another, the line in question passing through their point of contact. It follows from a straightforward calculation that on the line (1.6.14)

$$\left.\begin{aligned}[\operatorname{Im}\sqrt{(\sigma-\Lambda_{12})}]^2 &= -\frac{1}{2}\cos\frac{1}{2}\varphi\left(s-\frac{1}{2}l\cos\frac{1}{2}\varphi\right)+\frac{1}{4}l\sin^2\frac{1}{2}\varphi+\frac{1}{2}\left[\left(s-\frac{1}{2}l\cos\frac{1}{2}\varphi\right)^2+\frac{1}{4}l^2\sin^2\frac{1}{2}\varphi\right]^{\frac{1}{2}},\\ [\operatorname{Im}\sqrt{(\lambda-\sigma)}]^2 &= \frac{1}{2}\cos\frac{1}{2}\varphi\left(s-\frac{3}{2}l\cos\frac{1}{2}\varphi\right)+\frac{1}{4}l\sin^2\frac{1}{2}\varphi+\frac{1}{2}\left[\left(s-\frac{3}{2}l\cos\frac{1}{2}\varphi\right)^2+\frac{1}{4}l^2\sin^2\frac{1}{2}\varphi\right]^{\frac{1}{2}}.\end{aligned}\right\} \tag{1.6.15}$$

Substituting this into eq. (1.6.11) yields

$$I \leq \text{const.}\int_{-\infty}^{\infty}\left[t\cos\frac{1}{2}\varphi+\frac{1}{2}l\sin^2\frac{1}{2}\varphi+\left(t^2+\frac{1}{4}l^2\sin^2\frac{1}{2}\varphi\right)^{\frac{1}{2}}\right]^{-\frac{3}{2}}dt. \tag{1.6.16}$$

The change of variables $t=\frac{1}{2}l\sin\frac{1}{2}\varphi\sinh\psi$ now gives

$$I \leq \text{const.}\frac{1}{\sqrt{l}\sin^2\frac{1}{2}\varphi}\int_{-\infty}^{\infty}\cosh\psi\left[\frac{\cos\frac{1}{2}\varphi}{\sin\frac{1}{2}\varphi}\sinh\psi+\frac{1}{\sin\frac{1}{2}\varphi}\cosh\psi+1\right]^{-\frac{3}{2}}d\psi. \tag{1.6.17}$$

At this point it is convenient to write

$$\frac{\cos\frac{1}{2}\varphi}{\sin\frac{1}{2}\varphi}=\sinh\chi,\qquad \frac{1}{\sin\frac{1}{2}\varphi}=\cosh\chi,\qquad \omega=\psi+\chi, \tag{1.6.18}$$

and to go over from the integration variable ψ to ω. Then the inequality for I takes the form

$$I \leq \text{const.}\frac{1}{\sqrt{l}\sin^2\frac{1}{2}\varphi}\int_{-\infty}^{\infty}\left[\frac{1}{\sin\frac{1}{2}\varphi}\cosh\omega-\frac{\cos\frac{1}{2}\varphi}{\sin\frac{1}{2}\varphi}\sinh\omega\right](\cosh\omega+1)^{-\frac{3}{2}}d\omega. \tag{1.6.19}$$

Since the term involving $\sinh\omega$ clearly gives no contribution to the integral, we finally obtain

$$I \leq \text{const.}\frac{1}{\sqrt{l}\sin^3\frac{1}{2}\varphi}=\text{const.}\frac{|\lambda-\Lambda_{12}|}{[\operatorname{Im}\sqrt{(\lambda-\Lambda_{12})}]^3}. \tag{1.6.20}$$

This must now be combined with eq. (1.6.9) and with the last line of eq. (1.6.7). If in the latter expression we use the explicit coordinate-dependence of V_{13} and V_{23} as given by eq. (1.6.1), we get

$$\int |G_0^{(3)}(\boldsymbol{x},\boldsymbol{y};\lambda) - G_{12}^{(3)}(\boldsymbol{x},\boldsymbol{y};\lambda)|^2 \, |V_{13}(c_{13}^1\boldsymbol{y}_1 + c_{13}^2\boldsymbol{y}_2) + V_{23}(c_{23}^1\boldsymbol{y}_1 + c_{23}^2\boldsymbol{y}_2)|^2 d\boldsymbol{x}_1 \, d\boldsymbol{y}_1 \, d(\boldsymbol{x}_2 - \boldsymbol{y}_2) \, d\boldsymbol{y}_2 \Big| \leq \text{const.} \frac{|\lambda - \Lambda_{12}|^2}{[\text{Im} \sqrt{(\lambda - \Lambda_{12})}]^6} . \Big\} \quad (1.6.21)$$

It will be observed that, as regards the integration with respect to $\boldsymbol{y}_2$, it is essential that c_{13}^2 and c_{23}^2 do not vanish, and that V_{ij} is assumed to be square-integrable.

The inequality (1.6.21) obviously states that in the sum for $K_{123}^{(3)}$ the term with $i, j = 1, 2$ belongs to $\mathfrak{L}^2$ provided λ is an interior point of the complex plane cut from Λ_{12} to ∞. If for a moment we imagine a choice of coordinates adapted to $i, j = 1, 3$ or $2, 3$, we see that these cases can be discussed along exactly the same lines. Hence each term in the sum for $K_{123}^{(3)}$ belongs to $\mathfrak{L}^2$, and so does $K_{123}^{(3)}$ itself, provided λ is inside the complex plane cut from $\min(\Lambda_{12}, \Lambda_{13}, \Lambda_{23})$ to ∞. Moreover, it follows with eq. (1.6.21) that if λ is negative and sufficiently small, the $\mathfrak{L}^2$-norm of $K_{123}^{(3)}$ is less than 1. From this it follows by the argument used in connection with eqs. (1.3.49) and (1.3.50) that the spectrum of $H_{123}^{(3)}$ is bounded below.

Summarizing, we see that the properties of $K_{123}^{(3)}$ are such that in the λ-plane cut from $\min(\Lambda_{12}, \Lambda_{13}, \Lambda_{23})$ to ∞ we can evaluate the Green function $G_{123}^{(3)}$ by the Fredholm method outlined in section 1.3. In the cut plane the resolvent is regular except for possible poles confined to a finite interval of the real axis. Now it will be shown in a forthcoming paper on the theory of scattering that in the present case there is a continuous spectrum from $\min(\Lambda_{12}, \Lambda_{13}, \Lambda_{23})$ to ∞. Hence, if there are no poles in the cut plane and we introduce $\Lambda_{123} \equiv \min(\Lambda_{12}, \Lambda_{13}, \Lambda_{23})$, the Λ_{123} thus defined is the lower bound of the spectrum. And if there are poles, there is still a lower bound Λ_{123}, which is then less than $\min(\Lambda_{12}, \Lambda_{13}, \Lambda_{23})$. In either case it is clear from previous arguments that the norm of $R_{123}^{(3)}$ satisfies

$$\| R_{123}^{(3)}(\lambda) \| \leq \min\left(\frac{1}{[\text{Im} \sqrt{(\lambda - \Lambda_{123})}]^2}, \frac{1}{|\text{Im}\, \lambda|} \right), \quad (1.6.22)$$

cf. eqs. (1.3.29) and (1.4.36). Furthermore, with eq. (1.6.21) it is not difficult to see that

$$|K_{123}^{(3)}(\lambda)| \leq \text{const.} \frac{|\lambda - \Lambda_{123}|}{[\text{Im} \sqrt{(\lambda - \Lambda_{123})}]^3} . \quad (1.6.23)$$

It follows from the foregoing that in the expression (1.6.6) for the Green function $G_{123}^{(3)}$ the last term on the right-hand side belongs to $\mathfrak{L}^2$. Clearly it is the kernel of the integral operator

$$O \equiv R_{123}^{(3)} - \sum_{i<j} R_{ij}^{(3)} + 2 R_0^{(3)}. \quad (1.6.24)$$

But this operator is unique in the sense that, if f and g are any two functions in $\mathfrak{L}^2$, the quantity (g, Of) is uniquely determined. From this it follows with arguments such

as used in section 1.5.3 that, among all the kernels in $\mathfrak{L}^2$, the last term on the right-hand side of eq. (1.6.6) is unique. Also, the Green functions $G_0^{(3)}$ and $G_{ij}^{(3)}$ are unique in the sense specified in section 1.5.3. As a result, $G_{123}^{(3)}$ is unique in the sense that it can be written as a sum of functions with suitable integrability properties in only one way.

As a concluding remark it seems worth while to point out that in the above arguments it is an essential step to consider the somewhat complicated resolvent equation (1.6.5). If we had merely confined ourselves to eq. (1.6.2), we should have had a kernel of the form

$$G_0^{(3)}(\boldsymbol{x}, \boldsymbol{y}; \lambda) \sum_{i<j} V_{ij}(\boldsymbol{y}), \tag{1.6.25}$$

which certainly does not belong to $\mathfrak{L}^2$. For $G_0^{(3)}$ depends only on the difference $|\boldsymbol{x}-\boldsymbol{y}|$, according to eq. (1.5.25), and V_{12} depends only on $\boldsymbol{y}_1$. Hence $G_0^{(3)} V_{12}$ might be a square-integrable function of $\boldsymbol{x}_1-\boldsymbol{y}_1$, $\boldsymbol{x}_2-\boldsymbol{y}_2$, and $\boldsymbol{y}_1$ at best, rather than a function which is square-integrable with respect to four independent variables. However, even square-integrability with respect to three variables does not hold, since for $\boldsymbol{x} = \boldsymbol{y}$ the function $G_0^{(3)}$ has a singularity which prevents it from being square-integrable. Similar remarks apply to $G_0^{(3)} V_{13}$ and $G_0^{(3)} V_{23}$. Apparently the gist of eq. (1.6.5) is that in this equation a first approximation to $R_{123}^{(3)}$ is split off which is so good that the remainder is sufficiently regular and for large $|\boldsymbol{x}|$ and $|\boldsymbol{y}|$ falls off sufficiently rapidly for the equation to be soluble by the Fredholm method.

1.7. General numbers of particles

1.7.1. The resolvent equation

For discussing a general system of n particles, it is convenient to adopt the following notation. The symbol $(n, k)_{p(k)}$ denotes a certain way in which n particles can be split into k groups. Here the subscript $p(k)$ refers to the particular mode of division. Hence $p(k) = 1, 2, \ldots, N(n, k)$, where $N(n, k)$ denotes the number of ways in which n particles can be split into k groups. In the case of a particular division $(n, k)_{p(k)}$, let us imagine that all the interactions between the particles which together form a group are present, but that there are no interactions between particles belonging to different groups. The total interaction present in this way is denoted by $V(n, k)_{p(k)}$. The resolvent for the n-particle system with this interaction is called $R(n, k)_{p(k)}$. It will be shown that it is an integral operator with kernel $G(n, k)_{p(k)}$, the Green function. Comparing this notation with the one used thus far, we have, for example,

$$\left.\begin{aligned} V(n,1) &= \sum_{i<j} V_{ij} = V, & R(n,1) &= R_{12\ldots n}^{(n)} = R, \\ V(n,n) &= 0, & R(n,n) &= R_0^{(n)}. \end{aligned}\right\} \tag{1.7.1}$$

It is also convenient to make use of the symbol $\subset$, where $(n, k+1)_{p(k+1)} \subset (n, k)_{p(k)}$ indicates that the division $(n, k+1)_{p(k+1)}$ is contained in $(n, k)_{p(k)}$ in the sense that it can be obtained from $(n, k)_{p(k)}$ by splitting one group of the latter division into two, the remaining groups being left unchanged.

It is the purpose of the present section to show that for $n \geq 2$ the resolvent satisfies the equation

$$\left.\begin{aligned} R(n,1) &= \sum_{k=2}^{n} \sum_{p(k)=1}^{N(n,k)} (-1)^k (k-1)!\, R(n,k)_{p(k)} \\ &+ (-1)^{n-1} \sum_{\substack{p(n-1),\ldots,p(k),\ldots,p(2) \\ (n,n-1)_{p(n-1)} \subset \cdots \subset (n,k)_{p(k)} \subset \cdots \subset (n,2)_{p(2)}}} \ldots \sum \{ R(n,n) V(n,n-1)_{p(n-1)} R(n,n-1)_{p(n-1)} [V(n,n-2)_{p(n-2)} - V(n,n-1)_{p(n-1)}] \\ &\times \ldots R(n,k+1)_{p(k+1)} [V(n,k)_{p(k)} - V(n,k+1)_{p(k+1)}] \ldots R(n,2)_{p(2)} [V(n,1) - V(n,2)_{p(2)}] \} R(n,1). \end{aligned}\right\} \quad (1.7.2)$$

It will be observed that this equation was already established for $n = 2$ and $n = 3$. One of its features is that with all the groups of particles considered, there are always interactions between all the particles in each group. In other words, for discussing a mode of division in which the particles i, j, and l together form a group, we need the properties of the three-particle system with interaction $V_{ij} + V_{il} + V_{jl}$. But nowhere in the analysis does there occur an "unphysical" system such as the one with interaction $V_{il} + V_{jl}$ only.

In the notation of the preceding sections, eq. (1.7.2) gives an equation for $R^{(4)}_{1234}$ in terms of $R^{(4)}_0$, the six resolvents of the form $R^{(4)}_{ij}$, the four of the form $R^{(4)}_{ijk}$, and the three of the form $R^{(4)}_{ij;\,kl}$. Here $R^{(4)}_{ij;\,kl}$ stands for the resolvent of the four-particle system with interaction $V_{ij} + V_{kl}$ $(i \neq k,\ i \neq l,\ j \neq k,\ j \neq l)$. As will be discussed in some detail later, it can be obtained from a convolution of the resolvents $R^{(2)}_{ij}$, $R^{(2)}_{kl}$, and $R^{(2)}_0$.

Let us now assume that for $n-1$ particles $(n \geq 3)$ eq. (1.7.2) is established by a combinatorial argument based on the set of resolvent equations

$$R(n-1,k)_{p(k)} = R(n-1,l)_{p(l)} - R(n-1,l)_{p(l)} [V(n-1,k)_{p(k)} - V(n-1,l)_{p(l)}] R(n-1,k)_{p(k)} \quad (1.7.3)$$

with fixed $n-1$, the numbers $k, l, p(k), p(l)$ taking all possible values. Then it can be shown that the desired equation for n particles follows on combining resolvent equations of the form (1.7.3), but with $n-1$ replaced by n. For the proof it is convenient to choose a particular mode of division $(n, n-1)_{p(n-1)}$, and to consider the expression

$$\left.\begin{aligned} S(n,n-1)_{p(n-1)} \equiv (-1)^{n-1} &\sum_{\substack{p(n-2),\ldots,p(k),\ldots,p(2) \\ (n,n-1)_{p(n-1)} \subset (n,n-2)_{p(n-2)} \subset \cdots \subset (n,2)_{p(2)}}} \ldots \sum \{ R(n,n-1)_{p(n-1)} [V(n,n-2)_{p(n-2)} - V(n,n-1)_{p(n-1)}] \\ &\times \ldots R(n,2)_{p(2)} [V(n,1) - V(n,2)_{p(2)}] \} R(n,1), \end{aligned}\right\} \quad (1.7.4)$$

which differs from the sum in eq. (1.7.2) in that the factor $R(n,n)\,V(n,n-1)_{p(n-1)}$ and the summation over $p(n-1)$ are left out.

It is obvious that the division $(n,\,n-1)_{p(n-1)}$ must consist of $n-2$ free particles plus a group consisting of two particles, say the particles i and j. Furthermore, the structure of the sum in eq. (1.7.4) is such that in all factors the particles i and j always belong to the same group. Hence if we temporarily consider i and j together as one composite particle, we can say that in a certain sense the multiple sum in eq. (1.7.4) involves only $n-1$ (composite) particles. This suggests that we compare it with the multiple sum which appears in the equation for $n-1$ particles. By assumption, there is a combinatorial argument based on the set of equations (1.7.3) to show that the latter sum is equal to a certain linear combination of resolvents $R(n-1,\,k)_{p(k)}$. Now to each equation (1.7.3) there corresponds an equation

$$\left.\begin{aligned} R(n,k)_{p(k)} = {} & R(n,l)_{p(l)} - R(n,l)_{p(l)}\,[V(n,k)_{p(k)} - V(n,l)_{p(l)}]\,R(n,k)_{p(k)}, \\ & (n,n-1)_{p(n-1)} \subset (n,k)_{p(k)}, \\ & (n,n-1)_{p(n-1)} \subset (n,l)_{p(l)}, \end{aligned}\right\} \tag{1.7.5}$$

which obviously is of the same structure. If we imagine that in the $(n-1)$-particle system there is no particle j, and that the division $(n,k)_{p(k)}$ is obtained from $(n-1,\,k)_{p(k)}$ by adding j and attaching it to i, we can find $V(n,\,k)_{p(k)}$ from $V(n-1,\,k)_{p(k)}$ by merely replacing all terms V_{il} in the latter interaction by $V_{il}+V_{jl}$ and adding V_{ij}. The difference $V(n,k)_{p(k)} - V(n,l)_{p(l)}$ is obtained from $V(n-1,k)_{p(k)} - V(n-1,l)_{p(l)}$ by replacing all terms V_{im} in the latter expression by $V_{im}+V_{jm}$, the remaining terms being left unchanged. Summarizing, it is clear that with a suitable change of notation the combinatorial analysis which led to the $(n-1)$-particle equation can also be applied to the multiple sum in eq. (1.7.4). The result is that

$$S(n,n-1)_{p(n-1)} = -R(n,1) + \sum_{k=2}^{n-1} \sum_{\substack{p(k) \\ (n,n-1)_{p(n-1)} \subset (n,k)_{p(k)}}} (-1)^k (k-1)!\,R(n,k)_{p(k)}. \tag{1.7.6}$$

To this expression we must now apply $\sum_{p(n-1)} R(n,\,n)\,V(n,n-1)_{p(n-1)}$. In doing so, it is convenient to perform the summation with respect to $p(n-1)$ first. This makes it possible to use the relation

$$\sum_{\substack{p(n-1) \\ (n,n-1)_{p(n-1)} \subset (n,k)_{p(k)}}} V(n,n-1)_{p(n-1)} = V(n,k)_{p(k)}, \tag{1.7.7}$$

which expresses the fact that if we sum over all two-particle interactions V_{lm} subject to the condition that we only include interactions present in the division $(n,\,k)_{p(k)}$, we get the total $V(n,\,k)_{p(k)}$. Besides eq. (1.7.7), we need the relation

$$R(n,n)V(n,k)_{p(k)}R(n,k)_{p(k)} = R(n,n) - R(n,k)_{p(k)}. \tag{1.7.8}$$

This yields

$$\left.\begin{aligned} &\sum_{p(n-1)} R(n,n)V(n,n-1)_{p(n-1)}S(n,n-1)_{p(n-1)} \\ &= -R(n,n)V(n,1)R(n,1)+\sum_{k=2}^{n-1}\sum_{p(k)}(-1)^k(k-1)!\,R(n,n)V(n,k)_{p(k)}R(n,k)_{p(k)} \\ &= R(n,1)-R(n,n)+\sum_{k=2}^{n-1}\sum_{p(k)}(-1)^k(k-1)!\,R(n,n)-\sum_{k=2}^{n-1}\sum_{p(k)}(-1)^k(k-1)!R(n,k)_{p(k)}, \end{aligned}\right\} \quad (1.7.9)$$

where we recall that it was assumed that $n \geq 3$. This equation is almost equivalent to the desired relation (1.7.2). It only remains to show that

$$\left[1-\sum_{k=2}^{n-1}\sum_{p(k)}(-1)^k(k-1)!\right]R(n,n) = (-1)^n(n-1)!\,R(n,n) \quad (n\geq 3). \quad (1.7.10)$$

To check this, we note that $p(k)$ runs through $N(n,k)$ values, $N(n,k)$ denoting the number of ways in which n particles can be split into k groups. Hence what we want to show is that

$$1-\sum_{k=2}^{n-1}(-1)^k(k-1)!\,N(n,k) = (-1)^n(n-1)! \quad (n\geq 3). \quad (1.7.11)$$

Now any division $(n,k)_{p(k)}$ in which particle n belongs to a group of two or more particles can be obtained from a division $(n-1,k)_{p(k)}$ of the particles $1, 2, \ldots, n-1$ by adding particle n to one of the k groups. And each process of this sort yields a $(n,k)_{p(k)}$. Furthermore, any division $(n,k)_{p(k)}$ in which n forms a group by itself is obtained if to the corresponding division $(n-1,k-1)_{p(k-1)}$ particle n is added as a separate group. Hence

$$N(n,k) = kN(n-1,k)+N(n-1,k-1) \quad (n\geq 2, k\geq 2). \quad (1.7.12)$$

From this it follows that

$$\left.\begin{aligned} &1-\sum_{k=2}^{n-1}(-1)^k(k-1)!\,N(n,k) = 1-\sum_{k=2}^{n-1}(-1)^k k!\,N(n-1,k)-\sum_{k=2}^{n-1}(-1)^k(k-1)!\,N(n-1,k-1) \\ &= 1-(-1)^{n-1}(n-1)!\,N(n-1,n-1)-1!\,N(n-1,1) = (-1)^n(n-1)! \quad (n\geq 3), \end{aligned}\right\} \quad (1.7.13)$$

as we wanted to show.

Starting from the assumption that for $n-1$ particles eq. (1.7.2) had been established by combining various resolvent equations of the form (1.7.3), we have now shown that eq. (1.7.2) also holds true for n particles. In the course of the proof, we only had to combine the sets of resolvent equations (1.7.5) and (1.7.8). Hence what applied to $n-1$ before the proof has now been carried over to n, and we can proceed to $n+1$.

1.7.2. Auxiliary formulas

The reason for using the complicated resolvent equation (1.7.2) is that the operator which on its right-hand side acts on $R(n, 1)$ belongs to $\mathfrak{L}^2$. As a matter of fact, it is discussed in section 1.7.4 that even each separate term in the multiple sum belongs to $\mathfrak{L}^2$. However, before passing on to this point it is useful to start with a few auxiliary formulas.

The first of these concerns the result due to KATO(5) that there are positive constants α and β such that for every f in $\mathfrak{D}(H_0)$

$$\sum_{i<j} || V_{ij} f || < \alpha || H_0 f || + \beta || f ||, \tag{1.7.14}$$

where we may choose α as small as we like. From eq. (1.7.14) it follows in particular that

$$\left.\begin{aligned} || V R_0(\lambda) f || &\leq \alpha || H_0 R_0(\lambda) f || + \beta || R_0(\lambda) f || \\ &\leq \alpha || f || + \alpha |\lambda| \, || R_0(\lambda) f || + \beta || R_0(\lambda) f || \leq \left[\alpha + \frac{\alpha |\lambda| + \beta}{(\operatorname{Im} \sqrt{\lambda})^2}\right] || f ||. \end{aligned}\right\} \tag{1.7.15}$$

Hence if λ is negative and sufficiently small, the norm $|| V R_0(\lambda) ||$ is less than 1. Then the series for $(g, R(\lambda) f)$ in eq. (1.3.49) is convergent, so that the spectrum of H is bounded below. This argument applies to any number of particles.

We know from eq. (1.2.9) that $R(\lambda) f$ belongs to $\mathfrak{D}(H_0)$ for every f in $\mathfrak{L}^2$. Also,

$$H_0 R(\lambda) f = -\sum_{i<j} V_{ij} R(\lambda) f + \lambda R(\lambda) f + f, \tag{1.7.16}$$

so that by Minkowski's inequality

$$|| H_0 R(\lambda) f || \leq \sum_{i<j} || V_{ij} R(\lambda) f || + |\lambda| \, || R(\lambda) f || + || f || \leq \alpha || H_0 R(\lambda) f || + \left[\frac{\beta + |\lambda|}{[\operatorname{Im} \sqrt{(\lambda - \Lambda)}]^2} + 1\right] || f ||, \tag{1.7.17}$$

Λ being the lower bound of the spectrum of H. If we now choose $\alpha < 1$ and subtract $\alpha || H_0 R(\lambda) f ||$ from both sides of eq. (1.7.17), we find with eq. (1.7.14) that

$$\sum_{i<j} || V_{ij} R(\lambda) f || \leq \frac{\alpha}{1 - \alpha} \left[\frac{\beta + |\lambda|}{[\operatorname{Im} \sqrt{(\lambda - \Lambda)}]^2} + 1\right] || f || + \frac{\beta}{[\operatorname{Im} \sqrt{(\lambda - \Lambda)}]^2} || f ||. \tag{1.7.18}$$

From this it follows that the norm $|| V_{ij} R(\lambda) ||$ is uniformly bounded in any region

$$0 < \varepsilon \leq |\lambda - \Lambda|, \qquad 0 < \delta \leq \arg(\lambda - \Lambda) \leq 2\pi - \delta. \tag{1.7.19}$$

This is the required result.

For the following we also need several formulas concerning resolvents of the type $R(n, k)_{p(k)}$. Let us assume in particular that $k = 2$. Then the simplest situation arises if the division $(n, 2)_{p(2)}$ consists of a group of $n - 1$ particles plus a single

particle. In that case it is convenient to choose $n-2$ coordinates $\boldsymbol{x}$ for the internal motion of the $n-1$ particles, plus a coordinate $\boldsymbol{x}_3$ for the distance between the last particle and the centre of mass of the previous ones. If the groups consist of n_1 and n_2 particles with $n_1 \geq 2$, $n_2 \geq 2$, we use internal coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ within the respective groups, plus a coordinate $\boldsymbol{x}_3$ for the distance between the centres of mass of the groups. In the following we mainly concentrate on the latter case.

For discussing the total Hamiltonian $H(n, 2)_{p(2)}$, we have to consider the Hamiltonians $H(\boldsymbol{x}_1)$, $H(\boldsymbol{x}_2)$, and $H_0(\boldsymbol{x}_3)$ for the internal motion of the groups and their relative motion, respectively. In the first instance, these are defined as self-adjoint operators in the spaces $\mathfrak{L}^2(\boldsymbol{x}_1)$, $\mathfrak{L}^2(\boldsymbol{x}_2)$, and $\mathfrak{L}^2(\boldsymbol{x}_3)$. It was shown by KATO(5) that the domain of $H(\boldsymbol{x}_1)$ consists of all functions $f(\boldsymbol{x}_1)$ in $\mathfrak{L}^2(\boldsymbol{x}_1)$ for which $|\boldsymbol{k}_1|^2 \hat{f}(\boldsymbol{k}_1)$ belongs to $\mathfrak{L}^2(\boldsymbol{k}_1)$, $\hat{f}$ being the Fourier transform of f. With Kato's result, it is not difficult to see that $H(\boldsymbol{x}_1)$ can also be considered as a self-adjoint operator in the space $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, its domain being the set of functions $f(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ for which $|\boldsymbol{k}_1|^2 \hat{f}(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$ belongs to $\mathfrak{L}^2(\boldsymbol{k}_1, \boldsymbol{k}_2, \boldsymbol{k}_3)$. And similarly for $H(\boldsymbol{x}_2)$ and $H_0(\boldsymbol{x}_3)$. With this interpretation of the Hamiltonians, we have

$$H(n, 2)_{p(2)} \equiv H(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = H(\boldsymbol{x}_1) + H(\boldsymbol{x}_2) + H_0(\boldsymbol{x}_3). \tag{1.7.20}$$

Likewise, we can consider the operator $H(\boldsymbol{x}_1) + H(\boldsymbol{x}_2)$, say in the space $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$. It is of a slightly different structure from the Hamiltonians considered thus far, in that it is not quite the Hamiltonian of a system of particles with two-body interactions. But by Kato's(5) results, it is self-adjoint all the same. Also, it satisfies the crucial relations (1.2.9) and (1.7.14). From this it is easily deduced that the spectrum of $H(\boldsymbol{x}_1) + H(\boldsymbol{x}_2)$ is bounded below, and that there is a resolvent $R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ such that all the operators of the form $V_{ij}(\boldsymbol{x}_1) R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ or $V_{ij}(\boldsymbol{x}_2) R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ are regular and bounded uniformly in λ in a region of the form (1.7.19).

According to eq. (1.5.4), the resolvent $R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda)$ satisfies

$$(g(\boldsymbol{x}_1, \boldsymbol{x}_2), R(\boldsymbol{x}_1, \boldsymbol{x}_2; \lambda) f(\boldsymbol{x}_1, \boldsymbol{x}_2)) = \frac{1}{2\pi i} \oint_C (g(\boldsymbol{x}_1, \boldsymbol{x}_2), R(\boldsymbol{x}_2; \lambda - \sigma) R(\boldsymbol{x}_1; \sigma) f(\boldsymbol{x}_1, \boldsymbol{x}_2)) d\sigma. \tag{1.7.21}$$

It is convenient to denote this relationship by

$$R(\boldsymbol{x}_1, \boldsymbol{x}_2) = R(\boldsymbol{x}_1) * R(\boldsymbol{x}_2). \tag{1.7.22}$$

With this notation we have

$$[R(\boldsymbol{x}_1) * R(\boldsymbol{x}_2)] * R_0(\boldsymbol{x}_3) = R(\boldsymbol{x}_1) * [R(\boldsymbol{x}_2) * R_0(\boldsymbol{x}_3)]. \tag{1.7.23}$$

For either side yields the resolvent of $H(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, which is unique. Hence omitting the square brackets we may write

$$R(n, 2)_{p(2)} \equiv R(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3) = R(\boldsymbol{x}_1) * R(\boldsymbol{x}_2) * R_0(\boldsymbol{x}_3). \tag{1.7.24}$$

It is not difficult to see that in the case of a general division $(n, k)_{p(k)}$ this result can be extended by introducing k sets of internal coordinates for the motion within the k groups, $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_k$, plus a set of $k-1$ three-dimensional coordinates for the relative motion of the groups with respect to each other, $\boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{2k-1}$. As a generalization of eq. (1.7.24) this yields

$$R(n,k)_{p(k)} \equiv R(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \boldsymbol{x}_{k+1}, \ldots, \boldsymbol{x}_{2k-1}) = R(\boldsymbol{x}_1) * \ldots * R(\boldsymbol{x}_k) * R_0(\boldsymbol{x}_{k+1}) * \ldots * R_0(\boldsymbol{x}_{2k-1}). \quad (1.7.25)$$

It can be shown with the methods outlined in section 1.3.3 and at the beginning of the present section that all operators $V_{ij} R(n, k)_{p(k)}$ are regular and uniformly bounded in regions of the form (1.7.19). The same applies to operators such as

$$\left.\begin{aligned} & V_{ij}(\boldsymbol{x}_l)\, R(\boldsymbol{x}_1) * \ldots * R(\boldsymbol{x}_k) * R_0(\boldsymbol{x}_p) * \ldots * R_0(\boldsymbol{x}_q) \\ & (l = 1, 2, \ldots, k; \quad k+1 \leq p, \ldots, q \leq 2k-1). \end{aligned}\right\} \quad (1.7.26)$$

We now want to consider operators of the form

$$Q \equiv [R_b(\boldsymbol{x}) * R(\boldsymbol{y})]\, V(\boldsymbol{x})\, [R_a(\boldsymbol{x}) * R(\boldsymbol{y})], \quad (1.7.27)$$

where R_a, R_b, and R may or may not be different resolvents. It is essential that both operators $R(\boldsymbol{y})$ are the same. Also, $\| V(\boldsymbol{x})\, R_p(\boldsymbol{x}) \|$ $(p = a, b)$ must be bounded uniformly in a region of the form (1.7.19) and $\| V(\boldsymbol{x})\, [R_p(\boldsymbol{x}) * R(\boldsymbol{y})] \|$ must be bounded. It is assumed that the respective spectra are bounded below, their lower bounds being denoted by Λ_a, Λ_b, and Λ.

Writing out eq. (1.7.27) and using eq. (1.2.11), we get

$$\left.\begin{aligned} (g, Qf) &= -\frac{1}{4\pi^2} \int_C d\sigma \int_D d\tau\, (g(\boldsymbol{x},\boldsymbol{y}), R_b(\boldsymbol{x};\tau)\, R(\boldsymbol{y};\lambda-\tau)\, V(\boldsymbol{x})\, R_a(\boldsymbol{x};\sigma)\, R(\boldsymbol{y};\lambda-\sigma)\, f(\boldsymbol{x},\boldsymbol{y})) \\ &= -\frac{1}{4\pi^2} \int_C d\sigma \int_D \frac{d\tau}{\sigma-\tau} (g(\boldsymbol{x},\boldsymbol{y}), R_b(\boldsymbol{x};\tau) V(\boldsymbol{x}) R_a(\boldsymbol{x};\sigma) [R(\boldsymbol{y};\lambda-\tau) - R(\boldsymbol{y};\lambda-\sigma)] f(\boldsymbol{x},\boldsymbol{y})). \end{aligned}\right\} \quad (1.7.28)$$

At this point it is convenient to choose as the contours C and D the straight lines

$$\left.\begin{aligned} \sigma &= s e^{i\psi} - s_0 \quad (-\infty < s < \infty), \\ \tau &= t e^{i\psi} - t_0 \quad (-\infty < t < \infty), \end{aligned}\right\} \quad (1.7.29)$$

where we must make sure that

$$-s_0 < \Lambda_a, \quad -t_0 < \Lambda_b, \quad 0 < \delta \leq \psi \leq \pi - \delta. \quad (1.7.30)$$

It is obvious that in choosing ψ we must also take into account the location of the singularities of $R(\boldsymbol{y};\lambda-\sigma)$ and $R(\boldsymbol{y};\lambda-\tau)$. Furthermore, we take $-t_0 < -s_0$, so that the contour D is on the left of C.

It is now easy to evaluate the term on the right-hand side of eq. (1.7.28) which involves $R(\boldsymbol{y}; \lambda - \sigma)$. For the function

$$h(\boldsymbol{x}, \boldsymbol{y}) \equiv V(\boldsymbol{x})\, R_a(\boldsymbol{x}; \sigma)\, R(\boldsymbol{y}; \lambda - \sigma)\, f(\boldsymbol{x}, \boldsymbol{y}) \tag{1.731}$$

does not depend on τ and it belongs to $\mathfrak{L}^2(\boldsymbol{x}, \boldsymbol{y})$. Hence if we consider the expression

$$\int_D \frac{d\tau}{\sigma - \tau} (g(\boldsymbol{x}, \boldsymbol{y}), R_b(\boldsymbol{x}; \tau)\, h(\boldsymbol{x}, \boldsymbol{y})), \tag{1.7.32}$$

we see that its integrand is a regular function of τ in the half-plane on the left of D. Since in this half-plane the integrand tends to zero faster than $1/|\tau|$ when $|\tau|$ tends to ∞, it follows that the integral (1.7.32) vanishes.

To evaluate the remaining term in eq. (1.7.28), we want to invert the order of integration. Performing the integration with respect to σ first, we get a contributon from the pole $\sigma = \tau$ on the left of C. Hence in the abbreviated notation (1.7.22), the final result takes the form

$$[R_b(\boldsymbol{x}) * R(\boldsymbol{y})]\, V(\boldsymbol{x})\, [R_a(\boldsymbol{x}) * R(\boldsymbol{y})] = [R_b(\boldsymbol{x})\, V(\boldsymbol{x})\, R_a(\boldsymbol{x})] * R(\boldsymbol{y}). \tag{1.7.33}$$

It remains to justify the inversion of the order of integration. To check this, we observe that on D the norm $\| V(\boldsymbol{x})\, R_b(\boldsymbol{x}; \tau) \|$ is bounded uniformly in τ, by assumption. Furthermore,

$$\| R_a(\boldsymbol{x}; \sigma) \| \leq \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^2}, \quad \| R(\boldsymbol{y}; \lambda - \tau) \| \leq \frac{1}{[\operatorname{Im} \sqrt{(\lambda - \tau - \Lambda)}]^2}. \tag{1.7.34}$$

Hence to be able to apply the theorem on the inversion of Riemann integrals (Titchmarsh(15) section 1.85) it suffices to show that

$$I \equiv \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \frac{1}{|\sigma - \tau|} \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^2} \frac{1}{[\operatorname{Im} \sqrt{(\lambda - \tau - \Lambda)}]^2} < \infty, \tag{1.7.35}$$

and that the integrals

$$\left.\begin{aligned} I_s &\equiv \int_{-\infty}^{\infty} \frac{1}{|\sigma - \tau|} \frac{1}{[\operatorname{Im} \sqrt{(\lambda - \tau - \Lambda)}]^2} dt, \\ I_t &\equiv \int_{-\infty}^{\infty} \frac{1}{|\sigma - \tau|} \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^2} ds \end{aligned}\right\} \tag{1.7.36}$$

converge uniformly with respect to s and t, respectively.

As for I, it follows from Hölder's inequality that

$$I \leq \left[\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{1}{|\sigma-\tau|^{\frac{3}{2}}} \, \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^3} \right]^{\frac{1}{3}} \left[\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{1}{|\sigma-\tau|^{\frac{3}{2}}} \, \frac{1}{[\operatorname{Im} \sqrt{(\lambda - \tau - \Lambda)}]^3} \right]^{\frac{1}{3}} \times \left[\int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} dt \, \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^3} \, \frac{1}{[\operatorname{Im} \sqrt{(\lambda - \tau - \Lambda)}]^3} \right]^{\frac{1}{3}}. \tag{1.7.37}$$

Now it is easily checked that $\int_{-\infty}^{\infty} |\sigma - \tau|^{-\frac{3}{2}} \, dt$ is bounded uniformly in s. Also $\int_{-\infty}^{\infty} [\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^{-3} \, ds$ is convergent by the argument presented in section 1.6.2. Hence the first factor in eq. (1.7.37) is finite, and similarly for the remaining factors. Hence eq. (1.7.35) is satisfied.

In connection with I_t we have

$$\int_T^{\infty} \frac{1}{|\sigma-\tau|} \, \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^2} \, ds \leq \left[\int_{-\infty}^{\infty} \frac{1}{|\sigma-\tau|^2} \, ds \right]^{\frac{1}{2}} \left[\int_T^{\infty} \frac{1}{[\operatorname{Im} \sqrt{(\sigma - \Lambda_a)}]^4} \, ds \right]^{\frac{1}{2}}, \tag{1.7.38}$$

which clearly demonstrates the required uniform convergence. This completes the proof of eq. (1.7.33).

The proof of eq. (1.7.33) can easily be extended to show that

$$\begin{aligned} &[R_c(\boldsymbol{x}) * R_q(\boldsymbol{y})] V(\boldsymbol{x}) \{ [R_b(\boldsymbol{x}) V(\boldsymbol{x}) R_a(\boldsymbol{x})] * [R_q(\boldsymbol{y}) V(\boldsymbol{y}) R_p(\boldsymbol{y})] \} \\ &\quad = [R_c(\boldsymbol{x}) V(\boldsymbol{x}) R_b(\boldsymbol{x}) V(\boldsymbol{x}) R_a(\boldsymbol{x})] * [R_q(\boldsymbol{y}) V(\boldsymbol{y}) R_p(\boldsymbol{y})], \end{aligned} \tag{1.7.39}$$

and similarly for more complicated cases. The essential point about eq. (1.7.39) is that it contains a product $R_q(\boldsymbol{y}) \, R_q(\boldsymbol{y})$ which is reduced to a single factor $R_q(\boldsymbol{y})$. In the course of the proof, it must be assumed that various operators of the form VR are suitably bounded. It is not difficult to see that this condition is fulfilled in all cases in which eq. (1.7.39) is used in the next section.

1.7.3. An analysis of the multiple sum

We are now in a position to study the operator which on the right-hand side of eq. (1.7.2) acts on $R(n, 1)$. Let us choose a particular term of the multiple sum. This is characterized by a certain set $p(n-1), p(n-2), \ldots, p(2)$. Let us further assume that the division $(n, 2)_{p(2)}$ in question consists of two groups of at least two particles each. Then it is convenient to use internal coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ and a relative coordinate $\boldsymbol{x}_3$ as in section 1.7.2. With this choice of coordinates, eqs. (1.7.22) and (1.7.24) show that

$$R(n, 2)_{p(2)} = R(\boldsymbol{x}_1, \boldsymbol{x}_2) * R_0(\boldsymbol{x}_3). \tag{1.7.40}$$

Since in the expression we are considering $(n,3)_{p(3)} \subset (n,2)_{p(2)}$, the operator $R(n,3)_{p(3)}$ $[V(n,2)_{p(2)} - V(n,3)_{p(3)}]$ which acts on $R(n,2)_{p(2)}$ refers to a further splitting of the two groups of particles. It does not change the situation with respect to $\boldsymbol{x}_3$. Hence the function $V(n,2)_{p(2)} - V(n,3)_{p(3)}$ depends on $\boldsymbol{x}_1, \boldsymbol{x}_2$ but not on $\boldsymbol{x}_3$, and we have essentially

$$R(n,3)_{p(3)}[V(n,2)_{p(2)} - V(n,3)_{p(3)}] = [R(\boldsymbol{x}_1,\boldsymbol{x}_2) * R_0(\boldsymbol{x}_3)]V(\boldsymbol{x}_1,\boldsymbol{x}_2), \quad (1.7.41)$$

where $R(\boldsymbol{x}_1, \boldsymbol{x}_2)$ is some resolvent, not the same as in eq. (1.7.40). Since the operators $R_0(\boldsymbol{x}_3)$ in eqs. (1.7.40) and (1.7.41) are the same, viz. the resolvent for two particles without interaction, we can apply eq. (1.7.33) with the result that

$$\left.\begin{aligned} Q \equiv R(n,3)_{p(3)}[V(n,2)_{p(2)} - V(n,3)_{p(3)}]R(n,2)_{p(2)} \\ = [R(\boldsymbol{x}_1,\boldsymbol{x}_2)V(\boldsymbol{x}_1,\boldsymbol{x}_2)R(\boldsymbol{x}_1,\boldsymbol{x}_2)] * R_0(\boldsymbol{x}_3). \end{aligned}\right\} \quad (1.7.42)$$

With the help of eq. (1.7.39) the argument can be extended to the remaining factors in eq. (1.7.2). In the present symbolic notation this finally yields

$$\left.\begin{aligned} (-1)^{n-1}R(n,n)V(n,n-1)_{p(n-1)} \ldots R(n,2)_{p(2)} \\ = -[(-1)^n R(\boldsymbol{x}_1,\boldsymbol{x}_2)V(\boldsymbol{x}_1,\boldsymbol{x}_2) \ldots R(\boldsymbol{x}_1,\boldsymbol{x}_2)] * R_0(\boldsymbol{x}_3). \end{aligned}\right\} \quad (1.7.43)$$

We must now analyse the term in square brackets in eq. (1.7.43). Since each resolvent $R(\boldsymbol{x}_1, \boldsymbol{x}_2)$ refers to a system split into groups of particles with coordinates $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively, it can be written in the form of a convolution according to eq. (1.7.22). In particular, if in the division $(n,2)_{p(2)}$ there are n_1 particles with coordinates $\boldsymbol{x}_1$, and n_2 particles with coordinates $\boldsymbol{x}_2$, then the factor $R(\boldsymbol{x}_1, \boldsymbol{x}_2)$ which derives from $R(n,2)_{p(2)}$ can be obtained from a convolution of the resolvents for n_1 and n_2 particles with all their interactions. To bring this out in the notation, we write, with eq. (1.7.40),

$$R(n,2)_{p(2)} = R(\boldsymbol{x}_1; n_1, 1) * R(\boldsymbol{x}_2; n_2, 1) * R(\boldsymbol{x}_3; 2, 2), \quad (1.7.44)$$

where for instance $R(\boldsymbol{x}_3; 2, 2)$ is the resolvent for two particles divided into two groups, i.e. two particles without interaction, which acts on functions of $\boldsymbol{x}_3$.

To go over from the division $(n,2)_{p(2)}$ to $(n,3)_{p(3)} \subset (n,2)_{p(2)}$, it is necessary to split one of the two groups into two. Let this be the group of n_1 particles. Then the division $(n,3)_{p(3)}$ consists of a group of n_2 particles, plus a composite group of n_1 particles which itself consists of two subgroups according to a certain splitting $(n_1,2)_{q(2)}$. In the notation corresponding to eq. (1.7.44), this can be denoted by

$$R(n,3)_{p(3)} = R(\boldsymbol{x}_1; n_1, 2)_{q(2)} * R(\boldsymbol{x}_2; n_2, 1) * R(\boldsymbol{x}_3; 2, 2). \quad (1.7.45)$$

Also, the interaction $V(n,2)_{p(2)} - V(n,3)_{p(3)}$, which is the interaction between the two subgroups, is nothing but $V(\boldsymbol{x}_1; n_1, 1) - V(\boldsymbol{x}_1; n_1, 2)_{q(2)}$. Hence with eq. (1.7.42),

$$\left.\begin{aligned} Q = \{[R(\boldsymbol{x}_1; n_1, 2)_{q(2)} * R(\boldsymbol{x}_2; n_2, 1)]\,[V(\boldsymbol{x}_1; n_1, 1) - V(\boldsymbol{x}_1; n_1, 2)_{q(2)}] \\ \times [R(\boldsymbol{x}_1; n_1, 1) * R(\boldsymbol{x}_2; n_2, 1)]\} * R(\boldsymbol{x}_3; 2, 2). \end{aligned}\right\} \quad (1.7.46)$$

Now we can again apply eq. (1.7.33), with the result that

$$\left.\begin{aligned} Q = \{R(\boldsymbol{x}_1; n_1, 2)_{q(2)}\,[V(\boldsymbol{x}_1; n_1, 1) - V(\boldsymbol{x}_1; n_1, 2)_{q(2)}]\,R(\boldsymbol{x}_1; n_1, 1)\} \\ * R(\boldsymbol{x}_2; n_2, 1) * R(\boldsymbol{x}_3; 2, 2). \end{aligned}\right\} \quad (1.7.47)$$

The division $(n, 4)_{p(4)}$ either implies a further splitting among the n_1 particles, or it splits the group of n_2 particles. In either case it can be treated in the same way as $(n, 3)_{p(3)}$. Similar remarks apply to all further divisions. Since by the time we have reached the division (n, n), the two groups we began with are completely split up into single particles, it is not difficult to see that we finally obtain

$$\left.\begin{aligned} (-1)^{n-1} R(n, n) V(n, n-1)_{p(n-1)} \ldots R(n, 2)_{p(2)} \\ = -[(-1)^{n_1-1} R(n_1, n_1) V(n_1, n_1-1)_{q_1(n_1-1)} \ldots R(n_1, 1)] \\ * [(-1)^{n_2-1} R(n_2, n_2) V(n_2, n_2-1)_{q_2(n_2-1)} \ldots R(n_2, 1)] * R(2, 2), \\ (n, n-1)_{p(n-1)} \subset (n, n-2)_{p(n-2)} \subset \ldots \subset (n, 2)_{p(2)}, \\ (n_1, n_1-1)_{q_1(n_1-1)} \subset (n_1, n_1-2)_{q_1(n_1-2)} \subset \ldots \subset (n_1, 2)_{q_1(2)}, \\ (n_2, n_2-1)_{q_2(n_2-1)} \subset (n_2, n_2-2)_{q_2(n_2-2)} \subset \ldots \subset (n_2, 2)_{q_2(2)}. \end{aligned}\right\} \quad (1.7.48)$$

1.7.4. The kernel

If we compare eqs. (1.7.2) and (1.7.48), we observe that the expressions in square brackets on the right-hand side of eq. (1.7.48) are nothing but terms in the multiple sums in the equations for $R(n_1, 1)$ and $R(n_2, 1)$. Let us therefore assume that by a previous analysis these were shown to be integral operators with kernels of the form

$$F_q^{(m)}(\boldsymbol{x}, \boldsymbol{y}; \lambda) \equiv \int K_q^{(m)}(\boldsymbol{x}, \boldsymbol{z}; \lambda)\, G^{(m)}(\boldsymbol{z}, \boldsymbol{y}; \lambda)\, d\boldsymbol{z} \quad (m = n_1, n_2), \qquad (1.7.49)$$

where $G^{(m)}$ is the Green function corresponding to $R(m, 1)$ and $K_q^{(m)}$ is a kernel belonging to $\mathfrak{L}^2$. Let us also assume that it was established that

$$\left.\begin{aligned} \| R(\lambda; m, 1) \| \leq [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(m)})}]^{-2}, \\ | K_q^{(m)}(\lambda) | \leq \text{const.} \, | \lambda - \Lambda^{(m)} |^{m-2} \, [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(m)})}]^{-\frac{5}{2}m+\frac{9}{2}}, \end{aligned}\right\} \quad (1.7.50)$$

where $\Lambda^{(m)}$ is the lower bound of the spectrum of $H(m, 1)$. These assumptions are in agreement with what we know for $m = 2$ and $m = 3$. In particular, the bound for $| K_q^{(m)} |$ agrees with eqs. (1.3.4) and (1.6.23).

Under the above assumptions we get the equality

$$\left.\begin{aligned} &-(g,[(-1)^{n_1-1}R(n_1,n_1)V(n_1,n_1-1)_{q_1(n_1-1)}\ldots R(n_1,1)] \\ &*[(-1)^{n_2-1}R(n_2,n_2)V(n_2,n_2-1)_{q_2(n_2-1)}\ldots R(n_2,1)]\,f) \\ &=-\frac{1}{2\pi i}\int_C d\sigma\int\bar{g}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int F_{q_2}^{(n_2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\lambda-\sigma)\,F_{q_1}^{(n_1)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)\,f(\boldsymbol{y}_1,\boldsymbol{y}_2)\,d\boldsymbol{y}_1\,d\boldsymbol{y}_2 \end{aligned}\right\} \quad (1.7.51)$$

for any f and g in $\mathfrak{L}^2(\boldsymbol{x}_1,\boldsymbol{x}_2)$. It is now shown first of all that if λ is inside the complex plane cut from $\Lambda^{(n_1)}+\Lambda^{(n_2)}$ to ∞,

$$I\equiv\left\{\int d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int d\boldsymbol{y}_1\,d\boldsymbol{y}_2\left[\int_C|F_{q_2}^{(n_2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\lambda-\sigma)\,F_{q_1}^{(n_1)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)|\,d\sigma\right]^2\right\}^{\frac{1}{2}}<\infty. \quad (1.7.52)$$

From this it follows with Fubini's theorem that on the right-hand side of eq. (1.7.51) the integration with respect to σ may be performed first, provided we are willing to consider it as a Lebesgue integration. If this is done, we get an expression of the form

$$\left.\begin{aligned} &-\int\bar{g}(\boldsymbol{x}_1,\boldsymbol{x}_2)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\int[F_{q_1}^{(n_1)}*F_{q_2}^{(n_2)}](\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}_1,\boldsymbol{y}_2;\lambda)\,f(\boldsymbol{y}_1,\boldsymbol{y}_2)\,d\boldsymbol{y}_1\,d\boldsymbol{y}_2, \\ &[F_{q_1}^{(n_1)}*F_{q_2}^{(n_2)}](\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}_1,\boldsymbol{y}_2;\lambda)\equiv\frac{1}{2\pi i}\int_C F_{q_2}^{(n_2)}(\boldsymbol{x}_2,\boldsymbol{y}_2;\lambda-\sigma)\,F_{q_1}^{(n_1)}(\boldsymbol{x}_1,\boldsymbol{y}_1;\sigma)\,d\sigma. \end{aligned}\right\} \quad (1.7.53)$$

If eq. (1.7.52) holds true, the kernel defined in the second line of eq. (1.7.53) belongs to $\mathfrak{L}^2$, its $\mathfrak{L}^2$-norm not exceeding I.

To prove the assertion (1.7.52), we deduce from eq. (1.7.50) that

$$\left.\begin{aligned} &I\leq\text{const.}\int_C|\sigma-\Lambda^{(n_1)}|^{n_1-2}\,[\operatorname{Im}\sqrt{(\sigma-\Lambda^{(n_1)})}]^{-\frac{5}{2}n_1+\frac{5}{2}} \\ &\times|\lambda-\sigma-\Lambda^{(n_2)}|^{n_2-2}[\operatorname{Im}\sqrt{(\lambda-\sigma-\Lambda^{(n_2)})}]^{-\frac{5}{2}n_2+\frac{5}{2}}\,d\sigma. \end{aligned}\right\} \quad (1.7.54)$$

If we now write

$$\lambda-\Lambda^{(n_1)}-\Lambda^{(n_2)}=le^{i\varphi} \quad (1.7.55)$$

and for the contour C take the straight line

$$\sigma=se^{\frac{1}{2}i\varphi}-\frac{1}{2}l+\Lambda^{(n_1)}\quad(-\infty<s<\infty), \quad (1.7.56)$$

then the imaginary parts in eq. (1.7.54) take exactly the forms given on the right-hand sides of eq. (1.6.15). With these expressions, it is not difficult to check that

$$|\sigma-\Lambda^{(n_1)}|\,[\operatorname{Im}\sqrt{(\sigma-\Lambda^{(n_1)})}]^{-2}\leq 4\left(\sin\frac{1}{2}\varphi\right)^{-2}, \quad (1.7.57)$$

and that the same bound applies to the corresponding term with $\lambda-\sigma-\Lambda^{(n_2)}$. Inserting these results in eq. (1.7.54) yields with Hölder's inequality that

$$I \leq \text{const.}\left(\sin\frac{1}{2}\varphi\right)^{-2n+8}\int_{-\infty}^{\infty}\left[t\cos\frac{1}{2}\varphi+\frac{1}{2}l\sin^2\frac{1}{2}\varphi+\left(l^2+\frac{1}{4}l^2\sin^2\frac{1}{2}\varphi\right)^{\frac{1}{2}}\right]^{-\frac{1}{4}n-\frac{31}{2}}dt, \quad (1.7.58)$$

where we have written $n_1+n_2=n$. The reasoning applied to eq. (1.6.16) can now be used to show that

$$I \leq \text{const.}\,|\lambda-\Lambda^{(n_1)}-\Lambda^{(n_2)}|^{n-3}\,[\operatorname{Im}\sqrt{(\lambda-\Lambda^{(n_1)}-\Lambda^{(n_2)})}]^{-\frac{5}{2}n+5}, \quad (1.7.59)$$

so that eq. (1.7.52) is indeed satisfied.

The conclusion is thus far that if λ is inside the complex plane cut from $\Lambda^{(n_1)}+\Lambda^{(n_2)}$ to ∞, the operator on the left-hand side of eq. (1.7.51) is an integral operator which in the space $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2)$ belongs to $\mathfrak{L}^2$. According to eq. (1.7.48), the operator in question plays a prominent part in the equation for $R(n, 1)$, eq. (1.7.2). To discuss the latter equation more fully, particularly the term of the multiple sum we are considering in the present section, we note that the factor $V(n, 1)-V(n, 2)_{p(2)}$ stands for the interaction between the groups of n_1 and n_2 particles which together form the division $(n, 2)_{p(2)}$. Consequently, this factor depends in an essential way on the distance between the centres of mass of the two groups, which was denoted by $\boldsymbol{x}_3$. To bring this out explicitly, we note that since $\boldsymbol{x}_1$ describes the internal motion of n_1 particles, it must have n_1-1 three-dimensional components. Let us denote these by $\boldsymbol{x}_{1,r}$ $(r=1, 2, \ldots, n_1-1)$, and let us denote the components of $\boldsymbol{x}_2$ by $\boldsymbol{x}_{2,s}$ $(s=1, 2, \ldots, n_2-1)$. Then if we take into account all the two-particle interactions between the two groups considered, we get

$$V(n,1)-V(n,2)_{p(2)}=\sum_{i,j}V_{ij}\Big(\sum_r c_{ij}^{1,r}\boldsymbol{x}_{1,r}+\sum_s c_{ij}^{2,s}\boldsymbol{x}_{2,s}+c_{ij}^{3}\boldsymbol{x}_3\Big), \quad (1.7.60)$$

with certain coefficients c. The point is now that none of the coefficients c_{ij}^3 vanishes, owing to the meaning of the interactions V_{ij} involved in eq. (1.7.60). Hence if either side of eq. (1.7.60) is denoted by $V_p(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$, it follows that $V_p(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ belongs to $\mathfrak{L}^2(\boldsymbol{x}_3)$.

Combining this result with eqs. (1.7.48), (1.7.51), (1.7.53), we now consider

$$\left.\begin{aligned}&(g,(-1)^{n-1}R(n,n)V(n,n-1)_{p(n-1)}\ldots R(n,2)_{p(2)}[V(n,1)-V(n,2)_{p(2)}]f)\\&\quad=-\frac{1}{2\pi i}\int_D d\tau\int\bar{g}(\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{x}_3)\,d\boldsymbol{x}_1\,d\boldsymbol{x}_2\,d\boldsymbol{x}_3\int[F_{q_1}^{(n_1)}*F_{q_2}^{(n_2)}](\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}_1,\boldsymbol{y}_2;\tau)\\&\qquad\times G_0^{(2)}(\boldsymbol{x}_3,\boldsymbol{y}_3;\lambda-\tau)V_p(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)f(\boldsymbol{y}_1,\boldsymbol{y}_2,\boldsymbol{y}_3)\,d\boldsymbol{y}_1\,d\boldsymbol{y}_2\,d\boldsymbol{y}_3\end{aligned}\right\} \quad (1.7.61)$$

for any f and g in $\mathfrak{L}^2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$. From eqs. (1.6.10) and (1.7.59), it is clear that

$$\left.\begin{aligned}&J\equiv\Big\{\int d\boldsymbol{x}\int d\boldsymbol{y}\Big[\int_D|[F_{q_1}^{(n_1)}*F_{q_1}^{(n_2)}](\boldsymbol{x}_1,\boldsymbol{x}_2,\boldsymbol{y}_1,\boldsymbol{y}_2;\tau)\,G_0^{(2)}(\boldsymbol{x}_3,\boldsymbol{y}_3;\lambda-\tau)V_p(\boldsymbol{y})|\,d\tau\Big]^2\Big\}^{\frac{1}{2}}\\&\leq\text{const.}\int_D|\tau-\Lambda^{(n_1)}-\Lambda^{(n_2)}|^{n-3}\,[\operatorname{Im}\sqrt{(\tau-\Lambda^{(n_1)}-\Lambda^{(n_2)})}]^{-\frac{5}{2}n+5}\,[\operatorname{Im}\sqrt{(\lambda-\tau)}]^{-\frac{1}{2}}\,d\tau.\end{aligned}\right\} \quad (1.7.62)$$

By the method applied to I above, this can be further reduced to

$$J \leq \text{const.} \, |\lambda - \Lambda^{(n_1)} - \Lambda^{(n_2)}|^{n-2} \, [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(n_1)} - \Lambda^{(n_2)})}]^{-\frac{5}{2}n+\frac{9}{2}} . \tag{1.7.63}$$

Hence J is finite if λ is inside the complex plane cut from $\Lambda^{(n_1)} + \Lambda^{(n_2)}$ to ∞. Then in eq. (1.7.61) the integration with respect to τ may be performed first. And if we define

$$K_p^{(n)}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3, \boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3; \lambda)$$
$$\equiv -\frac{1}{2\pi i} \int_D [F_{q_1}^{(n_1)} * F_{q_2}^{(n_2)}] (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{y}_1, \boldsymbol{y}_2; \tau) \, G_0^{(2)}(\boldsymbol{x}_3, \boldsymbol{y}_3; \lambda - \tau) \, V_p(\boldsymbol{y}_1, \boldsymbol{y}_2, \boldsymbol{y}_3) \, d\tau , \tag{1.7.64}$$

it follows that the operator on the left-hand side of eq. (1.7.61) is an integral operator with kernel $K_p^{(n)}$. It belongs to $\mathfrak{L}^2$, and its $\mathfrak{L}^2$-norm does not exceed J.

The foregoing analysis applies to all the terms in the multiple sum in eq. (1.7.2) for which the division $(n, 2)_{p(2)}$ consists of two groups of at least two particles each. In the case of a division into a group of $n-1$ particles plus one single particle, we can confine ourselves to a set of $n-2$ internal coordinates $\boldsymbol{x}$ within the large group, plus a coordinate $\boldsymbol{x}_3$ for the distance between the centre of mass of this group and the last particle. We still want to use eq. (1.7.43), but we may skip equations (1.7.44) to (1.7.48) and (1.7.51) to (1.7.59). For it is easily seen that in the simple case in question the operator in square brackets in eq. (1.7.43) is a term of the multiple sum in the equation for $R(n-1, 1)$. Hence by assumption it is of the form $F_q^{(n-1)}(\boldsymbol{x}, \boldsymbol{y}; \lambda)$ given by eq. (1.7.49), with $K_q^{(n-1)}$ and $R(n-1, 1)$ satisfying eq. (1.7.50). According to this assumption, $F_q^{(n-1)}$ belongs to $\mathfrak{L}^2$, and

$$|F_q^{(n-1)}(\lambda)| \leq \text{const.} \, |\lambda - \Lambda^{(n-1)}|^{n-3} \, [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(n-1)})}]^{-\frac{5}{2}n+5} . \tag{1.7.65}$$

The argument applied to eq. (1.7.61) can now be used to show that corresponding to eq. (1.7.64) we get a kernel

$$K_p^{(n)}(\boldsymbol{x}, \boldsymbol{x}_3, \boldsymbol{y}, \boldsymbol{y}_3; \lambda) \equiv -\frac{1}{2\pi i} \int_D F_q^{(n-1)}(\boldsymbol{x}, \boldsymbol{y}; \tau) \, G_0^{(2)}(\boldsymbol{x}_3, \boldsymbol{y}_3; \lambda - \tau) \, V_p(\boldsymbol{y}, \boldsymbol{y}_3) \, d\tau , \tag{1.7.66}$$

the $\mathfrak{L}^2$-norm of which does not exceed

$$J \leq \text{const.} \, |\lambda - \Lambda^{(n-1)}|^{n-2} \, [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(n-1)})}]^{-\frac{5}{2}n+\frac{9}{2}} . \tag{1.7.67}$$

This is a bound of the same form as eq. (1.7.63), which is thus seen to apply to all the terms in the multiple sum of eq. (1.7.2).

The general result is therefore that the resolvent equation for n particles is equivalent to an equation of the form

$$\left.\begin{aligned} h(\boldsymbol{x};\lambda;n,1) = \sum_{k=2}^{n}\sum_{p(k)=1}^{N(n,k)}(-1)^k(k-1)!\,h(\boldsymbol{x};\lambda;n,k)_{p(k)} + \int \sum_p K_p^{(n)}(\boldsymbol{x},\boldsymbol{y};\lambda)\,h(\boldsymbol{y};\lambda;n,1)\,d\boldsymbol{y}, \\ h(\boldsymbol{x};\lambda;n,k)_{p(k)} \equiv R(\boldsymbol{x};\lambda;n,k)_{p(k)}\,f(\boldsymbol{x}), \end{aligned}\right\} \quad (1.7.68)$$

where $\boldsymbol{x}$ now stands for all the space coordinates. If for m particles ($m = 2, 3, \ldots, n-1$) the kernels $K_q^{(m)}$ and the Green function $G^{(m)}$ are known, the kernels $K_p^{(n)}$ in eq. (1.7.68) can be found with convolutions according to eqs. (1.7.49), (1.7.53), and (1.7.64), cq. (1.7.66). All the functions $h(\boldsymbol{x};\lambda;n,k)_{p(k)}$ with $k \geq 2$ can likewise be obtained from the Green functions for less than n particles. For it follows directly from eq. (1.7.25) that the inner product $(g, R(n,k)_{p(k)}\,f)$ can be evaluated by convolutions, and the argument used to get eq. (1.5.21) shows that this result can be carried over to the function $R(n,k)_{p(k)}\,f$ itself. Hence eq. (1.7.68) can be considered as an integral equation for the unknown function $h(\boldsymbol{x};\lambda;n,1)$. And since its kernel belongs to $\mathfrak{L}^2$, it can be solved and further analysed by the methods outlined in section 1.3.

It is obvious that in doing so we have to confine ourselves to the λ-plane cut from a suitable point $M^{(n)}$ to ∞. For in constructing the various kernels $K_p^{(n)}$, we had to observe cuts from $\Lambda^{(n_1)} + \Lambda^{(n_2)}$ to ∞, or from $\Lambda^{(n-1)}$ to ∞. And the convolutions by which we want to find the functions $R(n,k)_{p(k)}\,f$ with $k \geq 2$ necessitate similar cuts. Now the spectrum of $H(n,1)$ has a lower bound, say $\Lambda^{(n)}$. It is discussed in section 1.7.6 that this does not exceed $M^{(n)}$. In particular,

$$\Lambda^{(n)} \leq \Lambda^{(n_1)} + \Lambda^{(n_2)} \quad (n_1 + n_2 = n), \quad \Lambda^{(n)} \leq \Lambda^{(n-1)}. \quad (1.7.69)$$

Hence according to eqs. (1.7.63), (1.7.67) and (1.4.36),

$$\left.\begin{aligned} |K_p^{(n)}(\lambda)| \leq \text{const.}\,|\lambda - \Lambda^{(n)}|^{n-2}\,[\operatorname{Im}\sqrt{(\lambda-\Lambda^{(n)})}]^{-\frac{5}{2}n+\frac{9}{2}}, \\ \|R(\lambda;n,1)\| \leq [\operatorname{Im}\sqrt{(\lambda-\Lambda^{(n)})}]^{-2}, \end{aligned}\right\} \quad (1.7.70)$$

which is of exactly the same form as the assumed relations (1.7.50).

1.7.5. The Green function

It is the purpose of the present section to show that there is a Green function $G^{(n)} \equiv G(n,1)$ such that for every f in $\mathfrak{L}^2$

$$R(\boldsymbol{x};\lambda;n,1)\,f(\boldsymbol{x}) = \int G^{(n)}(\boldsymbol{x},\boldsymbol{y};\lambda)\,f(\boldsymbol{y})\,d\boldsymbol{y}. \quad (1.7.71)$$

Since the kernel of the resolvent equation belongs to $\mathfrak{L}^2$, it follows from the analysis used to obtain eq. (1.3.11) that to prove eq. (1.7.71), it is sufficient to show that there are Green functions for all the resolvents $R(n,k)_{p(k)}$ with $k \geq 2$, and to check that these satisfy symmetry relations analogous to eqs. (1.5.46) and (1.5.47).

To find an expression for the Green functions in question, we rewrite the resolvent equation in the symbolic form

$$R(n,1) = \sum c_k R(n_1,1) * R(n_2,1) * \ldots * R(n_k,1) * R_0 * \ldots * R_0 + K^{(n)} R(n,1), \quad (1.7.72)$$

where for the known resolvents $R(n,k)_{p(k)}$ with $k \geq 2$ we have used eq. (1.7.25). The c_k are certain constants, the summation is to be taken with respect to all possible divisions. It is obvious that in eq. (1.7.72) $n_i \leq n-1$ $(i = 1, 2, \ldots, k)$.

Let us now replace all the resolvents $R(n_i, 1)$ in eq. (1.7.72) by the right-hand sides of their resolvent equations. This yields a multiple sum a typical term of which is of the form

$$[K^{(n_1)} R(n_1,1)] * [K^{(n_2)} R(n_2,1)] * R(n_p,1) * \ldots * R(n_q,1) * R_0 * \ldots * R_0. \quad (1.7.73)$$

It contains a certain number, possibly 0, factors of the form KR, several factors R, and several factors R_0. In the factors R we have $n_p, \ldots, n_q \leq n-2$. To reduce the expression (1.7.73) further, we now replace the resolvents $R(n_p, 1), \ldots, R(n_q, 1)$ by the right-hand sides of their resolvent equations, and so on. In this way we finally obtain a multiple sum consisting of terms of the form

$$[K^{(n_1)} R(n_1,1)] * [K^{(n_2)} R(n_2,1)] * \ldots * [K^{(n_j)} R(n_j,1)] * R_0 * \ldots * R_0. \quad (1.7.74)$$

Here the convolutions may be performed in any desired order. For it was pointed out already in eq. (1.7.23) that this may be done in the case of three or more resolvents. And according to the resolvent equation, the quantities $K^{(n_i)} R(n_i, 1)$ are nothing but sums of resolvents.

Let us therefore begin with the multiple convolution $R_0 * \ldots * R_0$. According to eqs. (1.5.27) and (1.2.17), this yields a certain Hankel function,

$$G_0^{(n')}(\boldsymbol{x}_{j+1}, \boldsymbol{y}_{j+1}; \lambda) = \frac{i}{4}\left[\frac{\sqrt{\lambda}}{2\pi |\boldsymbol{x}_{j+1} - \boldsymbol{y}_{j+1}|}\right]^{\frac{3}{2}n'-\frac{5}{2}} H^{(1)}_{\frac{3}{2}n'-\frac{5}{2}}(\sqrt{\lambda}\,|\boldsymbol{x}_{j+1} - \boldsymbol{y}_{j+1}|), \quad (1.7.75)$$

where the number of dimensions of $\boldsymbol{x}_{j+1}$ and $\boldsymbol{y}_{j+1}$ is $3n'-3$. Obviously $n' \leq n$, the sign of equality applying only if there are no factors KR.

As for the operators KR, we assume as before that these are integral operators with kernels of the form F satisfying eqs. (1.7.49) and (1.7.50). If in general

$$K^{(n_i)} R(n_i,1) f(\boldsymbol{x}_i) = \int F^{(n_i)}(\boldsymbol{x}_i, \boldsymbol{y}_i; \lambda) f(\boldsymbol{y}_i)\, d\boldsymbol{y}_i, \quad (1.7.76)$$

where in the space $\mathfrak{L}^2(\boldsymbol{x}_i)$

$$|F^{(n_i)}(\lambda)|_{\boldsymbol{x}_i} \leq \text{const.}\, |\lambda - \Lambda^{(n_i)}|^{n_i-2} [\operatorname{Im} \sqrt{(\lambda - \Lambda^{(n_i)})}]^{-\frac{5}{2}n_i+\frac{5}{2}}, \quad (1.7.77)$$

then it follows immediately from the reasoning developed in the previous section that $(KR) * (KR)$ is an integral operator,

$$\left.\begin{aligned} &\{[K^{(n_i)}R(n_i,1)] * [K^{(n_j)}R(n_j,1)]\} f(\boldsymbol{x}_i,\boldsymbol{x}_j) \\ &= \int [F^{(n_i)} * F^{(n_j)}](\boldsymbol{x}_i,\boldsymbol{x}_j,\boldsymbol{y}_i,\boldsymbol{y}_j;\lambda) f(\boldsymbol{y}_i,\boldsymbol{y}_j)\, d\boldsymbol{y}_i\, d\boldsymbol{y}_j, \end{aligned}\right\} \quad (1.7.78)$$

where now in the space $\mathfrak{L}^2(\boldsymbol{x}_i,\boldsymbol{x}_j)$

$$|[F^{(n_i)} * F^{(n_j)}](\lambda)|_{\boldsymbol{x}_i,\boldsymbol{x}_j} \leq \text{const.}\,|\lambda - \Lambda^{(n_i)} - \Lambda^{(n_j)}|^{n_i+n_j-3}\,[\operatorname{Im}\sqrt{(\lambda-\Lambda^{(n_i)}-\Lambda^{(n_j)})}]^{-\frac{5}{2}(n_i+n_j)+5}. \quad (1.7.79)$$

This result can easily be extended to

$$\left.\begin{aligned} &\{[K^{(n_1)}R(n_1,1)] * [K^{(n_2)}R(n_2,1)] * \ldots * [K^{(n_j)}R(n_j,1)]\} f(\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_j) \\ &= \int [F^{(n_1)} * F^{(n_2)} * \ldots * F^{(n_j)}](\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_j,\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_j;\lambda) f(\boldsymbol{y}_1,\boldsymbol{y}_2,\ldots,\boldsymbol{y}_j)\, d\boldsymbol{y}_1\, d\boldsymbol{y}_2 \ldots d\boldsymbol{y}_j, \end{aligned}\right\} \quad (1.7.80)$$

the norm of the integral kernel satisfying

$$\left.\begin{aligned} &|[F^{(n_1)} * F^{(n_2)} * \ldots * F^{(n_j)}](\lambda)|_{\boldsymbol{x}_1,\boldsymbol{x}_2,\ldots,\boldsymbol{x}_j} \\ &\leq \text{const.}\,|\lambda - \sum_{i=1}^{j}\Lambda^{(n_i)}|^{n''-j-1}\,[\operatorname{Im}\sqrt{(\lambda-\sum_{i=1}^{j}\Lambda^{(n_i)})}]^{-\frac{5}{2}n''+\frac{5}{2}j},\quad n'' \equiv \sum_{i=1}^{j} n_i. \end{aligned}\right\} \quad (1.7.81)$$

To establish that the operator (1.7.74) is an integral operator, it remains to investigate

$$\left.\begin{aligned} &I \equiv \frac{1}{2\pi}\int_C d\sigma \int d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_j\, d\boldsymbol{x}_{j+1} \int d\boldsymbol{y}_1 \ldots d\boldsymbol{y}_j\, d\boldsymbol{y}_{j+1}\, |\,\bar{g}(\boldsymbol{x}_1,\ldots,\boldsymbol{x}_j,\boldsymbol{x}_{j+1}) \\ &\times [F^{(n_1)} * \ldots * F^{(n_j)}](\boldsymbol{x}_1,\ldots,\boldsymbol{x}_j,\boldsymbol{y}_1,\ldots,\boldsymbol{y}_j;\sigma)\, G_0^{(n')}(\boldsymbol{x}_{j+1},\boldsymbol{y}_{j+1};\lambda-\sigma) f(\boldsymbol{y}_1,\ldots,\boldsymbol{y}_j,\boldsymbol{y}_{j+1})|. \end{aligned}\right\} \quad (1.7.82)$$

For this we note that

$$\left.\begin{aligned} |\lambda|^{\frac{3}{4}n'-\frac{5}{4}} \int_0^\infty |H^{(1)}_{\frac{3}{2}n'-\frac{5}{2}}(r\sqrt{\lambda})|\, r^{\frac{3}{2}n'-\frac{3}{2}}\, dr &\leq |\lambda|^{\frac{3}{4}n'-\frac{3}{2}} (\operatorname{Im}\sqrt{\lambda})^{\frac{1}{2}} \left| \int_0^\infty H^{(1)}_{\frac{3}{2}n'-\frac{5}{2}}(ir\operatorname{Im}\sqrt{\lambda})\, r^{\frac{3}{2}n'-\frac{3}{2}}\, dr \right| \\ &\leq \text{const.}\,|\lambda|^{\frac{3}{4}n'-\frac{3}{2}} (\operatorname{Im}\sqrt{\lambda})^{-\frac{3}{2}n'+1}, \end{aligned}\right\} \quad (1.7.83)$$

where the first inequality follows from an integral representation of the Hankel function (WATSON(19) section 6.12), and the second one from evaluating the integral in the second member. With a change of variables as was made in eq. (1.5.34), it is now readily seen that

$$\left.\begin{aligned} I \leq \text{const.} \int_C d\sigma\, |\sigma - \sum_{i=1}^{j}\Lambda^{(n_i)}|^{n''-j-1}\,[\operatorname{Im}\sqrt{(\sigma-\sum_{i=1}^{j}\Lambda^{(n_i)})}]^{-\frac{5}{2}n''+\frac{5}{2}j} \\ \times |\lambda-\sigma|^{\frac{3}{4}n'-\frac{3}{2}}\,[\operatorname{Im}\sqrt{(\lambda-\sigma)}]^{-\frac{3}{2}n'+1}\,\|g\|\,\|f\|, \end{aligned}\right\} \quad (1.7.84)$$

which is finite provided C is a suitable contour and λ is inside the complex plane cut from $\sum_{i=1}^{j} \Lambda^{(n_i)}$ to ∞. Under these circumstances, the operator (1.7.74) can be written as an integral operator the kernel of which is of the form

$$\frac{1}{2\pi i}\int_C [F^{(n_1)} * \ldots * F^{(n_j)}](\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_j; \sigma)\, G_0^{(n')}(\boldsymbol{x}_{j+1}, \boldsymbol{y}_{j+1}; \lambda - \sigma)\, d\sigma. \quad (1.7.85)$$

Since for $k \geq 2$ the resolvents $R(n, k)_{p(k)}$ are linear combinations of operators of the form (1.7.74), it follows that there are Green functions $G(n, k)_{p(k)}$ which are linear combinations of functions of the form (1.7.85).

It is not difficult to see that the kernel given by eq. (1.7.85) depends only on $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_j$, and $\boldsymbol{x}_{j+1} - \boldsymbol{y}_{j+1}$. It satisfies

$$\frac{1}{2\pi}\int d\boldsymbol{x}_1 \ldots d\boldsymbol{x}_j \int d\boldsymbol{y}_1 \ldots d\boldsymbol{y}_j \Big[\int d(\boldsymbol{x}_{j+1} - \boldsymbol{y}_{j+1}) \Big| \int_C [F^{(n_1)} * \ldots * F^{(n_j)}](\boldsymbol{x}_1, \ldots, \boldsymbol{x}_j, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_j; \sigma) \times G_0^{(n')}(\boldsymbol{x}_{j+1}, \boldsymbol{y}_{j+1}; \lambda - \sigma)\, d\sigma \Big|^2 < \infty. \quad (1.7.86)$$

This is a situation analogous to the one discussed in section 1.5.3. With the methods outlined there, it is easily shown that the operator (1.7.74) can be written as an integral operator with a kernel satisfying eq. (1.7.86) in only one way. In this sense the kernel (1.7.85) is unique. Since the Green functions $G(n, k)_{p(k)}$ with $k \geq 2$ are linear combinations of functions of the form (1.7.85), they are likewise unique.

If for $m = 2, 3, \ldots, n-1$ the Green functions $G(m, k)_{q(k)}$ $(k = 1, 2, \ldots, m)$ satisfy symmetry relations of the forms (1.5.46) and (1.5.47), then so do the functions $F^{(m)}$, by the resolvent equation. With the argument used to get eqs. (1.5.46) and (1.5.47), it is then easily checked that similar symmetry relations apply to the functions $G(n, k)_{p(k)}$ $(k \geq 2)$. At this point the fact that the kernel $K^{(n)}$ belongs to $\mathfrak{L}^2$ can be used to show that there is a Green function $G^{(n)} \equiv G(n, 1)$ (cf. the proof of eq. (1.3.11)). Also, since the operator $K^{(n)} R(n, 1)$ belongs to $\mathfrak{L}^2$, its kernel is unique among all kernels in $\mathfrak{L}^2$. By the resolvent equation and the uniqueness of the functions $G(n, k)_{p(k)} (k \geq 2)$, this means that $G^{(n)}$ is unique in the sense that it can be written as a linear combination of functions with suitable integrability properties in only one way. It follows with the method of section 1.3.2 that $G^{(n)}$ also satisfies symmetry relations.

1.7.6. The spectrum

We conclude the present investigation with some general remarks on the spectrum of the Hamiltonian $H(n, 1)$. It was assumed in eqs. (1.7.49) and (1.7.50) that for $m = 2, 3, \ldots, n-1$ we know the kernels $K_q^{(m)}$ and the Green functions $G^{(m)}$ in the λ-plane cut from $\Lambda^{(m)}$ to ∞. Under this assumption we are able to evaluate the kernels $K_p^{(n)}$. However, it follows from eqs. (1.7.63) and (1.7.67) that in doing so we have to confine ourselves to the λ-plane cut from $\min(\Lambda^{(n_1)} + \Lambda^{(n_2)}, \Lambda^{(n-1)})$ to ∞, where the

minimum is to be taken with respect to all possible divisions of the n particles into two groups. Now it will be shown in a forthcoming paper on the theory of scattering that there is a continuous spectrum from $\min(A^{(n_1)}+A^{(n_2)}, A^{(n-1)})$ to ∞. From this it follows immediately that the lower bound $A^{(n)}$ of the spectrum of $H(n, 1)$ satisfies

$$A^{(n)} \leq \min(A^{(n_1)}+A^{(n_2)}, A^{(n-1)}) \quad (n_1+n_2=n). \tag{1.7.87}$$

This was already used in eq. (1.7.69).

Let us now consider the convolutions whereby we want to construct the Green functions $G(n, k)_{p(k)}$ with $k \geq 2$. According to eq. (1.7.84), these give rise to various cuts from $\sum_{i=1}^{j} A^{(n_i)}$ to ∞, where $\sum_{i=1}^{j} n_i = n'' \leq n$ $(2 \leq n_i \leq n-1)$. Now if $n'' \leq n-1$, eq. (1.7.87) shows that

$$\min(A^{(n-1)}) \leq \min(A^{(n'')}) \leq \sum_{i=1}^{j} A^{(n_i)}. \tag{1.7.88}$$

Also, if $n''=n$, the sum with respect to i must consist of at least two terms, since $n_i \leq n-1$. Hence in this case we may write

$$A^{(n-n_j)}+A^{(n_j)} \leq \sum_{i=1}^{j-1} A^{(n_i)}+A^{(n_j)} = \sum_{i=1}^{j} A^{(n_i)}, \tag{1.7.89}$$

where now $A^{(n-n_j)}$ refers to the particular system of $n-n_j$ particles which is obtained by putting together the groups of $n_1, n_2, \ldots, n_{j-1}$ particles which appear on the right-hand side of eq. (1.7.89), and taking into account all their interactions. It follows in any case that

$$\min(A^{(n_1)}+A^{(n_2)}, A^{(n-1)}) \leq \sum_{i=1}^{j} A^{(n_i)}. \tag{1.7.90}$$

Hence the cut owing to the kernel contains the cut caused by the construction of the Green functions $G(n, k)_{p(k)}$ with $k \geq 2$. If we define

$$M^{(n)} = \min(A^{(n_1)}+A^{(n_2)}, A^{(n-1)}) \quad (n_1+n_2=n), \tag{1.7.91}$$

the ultimate cut runs from $M^{(n)}$ to ∞. Now it is clear from the method of evaluating the Green function $G^{(n)}$ that $H(n, 1)$ cannot have a continuous spectrum beyond the interval $[M^{(n)}, \infty)$. Hence since there is a continuous spectrum from $M^{(n)}$ to ∞, as was remarked above, it follows that the continuous spectrum of $H(n, 1)$ coincides with the interval $[M^{(n)}, \infty)$. In the λ-plane cut from $M^{(n)}$ to ∞, there may be a discrete spectrum in a finite interval $[A^{(n)}, M^{(n)})$. It is not known if $M^{(n)}$ is a point of accumulation of the discrete spectrum (cf. the discussion at the end of section 1.3.4).

Summarizing, let us assume that for $m=2, 3, \ldots, n-1$ we know the kernels $K_q^{(m)}$ and the Green functions $G^{(m)}$ in the λ-plane cut from $A^{(m)}$ to ∞. Let the quantities $|K_q^{(m)}|$ and $||R(m, 1)||$ satisfy the inequalities (1.7.50). Then we can find the kernels $K_p^{(n)}$ and the Green functions $G(n, k)_{p(k)}$ $(k \geq 2)$ in the λ-plane cut from $M^{(n)}$ to ∞. The quantities $|K_p^{(n)}|$ satisfy the inequality (1.7.50). Also, if all the Green func-

tions $G(m, k)_{q(k)}$ satisfy symmetry relations of the forms (1.5.46) and (1.5.47), then so do the functions $G(n, k)_{p(k)}$ $(k \geq 2)$. Owing to this symmetry combined with the fact that each $|K_p^{(n)}|$ is finite, we can evaluate the Green function $G^{(n)} \equiv G(n, 1)$ for all values of λ not in the spectrum of $H(n, 1)$. In particular, $G^{(n)}$ can be found in the λ-plane cut from $\Lambda^{(n)}$ to ∞. It satisfies the symmetry relations (1.5.46) and (1.5.47). The norm $||R(n,1)||$ satisfies the inequality (1.7.50). We thus see that on the basis of our assumptions for $m = 2, 3, \ldots, n-1$ we can find all the relevant quantities for $m = n$. These satisfy the desired inequalities and symmetry relations. Also, we can evaluate the Green function $G^{(2)} \equiv G(2,1)$ from the known functions $K^{(2)}$ and $G(2,2)$. For $m = 2$ the inequalities and symmetry relations are satisfied. From this we may conclude that by iteration we are able to construct the Green function $G^{(n)}$ for any finite number n.

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